INTRODUCTION TO THE THEORY OF FOURIER'S SERIES

By Maxime Bôcher

The theory of Fourier's series, so interesting in itself and so important in its countless applications in pure and in applied mathematics, is treated at more or less length in almost all treatises on higher analysis. The problem mainly considered, and frequently considered to the exclusion of all others, is that of developing a given function in a Fourier's series and proving that, if the function is subjected to suitable restrictions, this development will actually converge to the value of the function. While this problem is fundamental, there are many other important questions connected with this subject. In the present treatment I have established the possibility of developing functions in Fourier's series by two different methods, the first (§§1–7) being built up around a central idea due to Poisson, while the second (§§11–13) is a modification of Dirichlet's classic proof. If the reader wishes to see how brief the treatment of this central problem can be made, he is advised to turn at once to §§11–12 which are entirely independent of the preceding sections. Around this question of developability I have grouped numerous other questions of greater or less importance in the theory, including references, at least, to some of the most recent contributions to the subject. I hope thus to have made it possible for readers to pursue the study of the subject further in such directions as they may select. In particular I should like to call attention to the elaboration contained in §9 of an interesting remark of Willard Gibbs concerning the nature of the convergence of a Fourier's series in the neighborhood of a point of discontinuity.

A side of the subject into which I have not gone to any extent is the question of obtaining extremely broad classes of functions whose Fourier's developments converge. In particular, with the exception of a few foot-notes in which the possibility of further extension is pointed out, I have restricted myself to the case, which alone occurs in the great majority of applications, in which the function to be developed has only a finite number of discontinuities.

It is useless to try to gain any real mastery of almost any part of higher

analysis without a firm grasp of the idea of uniform convergence of series,* and I have assumed familiarity with this idea and with the three or four fundamental theorems involved, on the part of the reader. Apart from this no knowledge beyond the elements of the calculus has been, in the main, assumed. Some readers, however, will find it desirable, at least on a first reading, to pass over a few of the more difficult formal proofs.

For a comprehensive account of the historical development of this and allied subjects the reader is referred to the monograph by H. Burkhardt: *Entwicklungen nach oscillirenden Functionen*, which is now in course of publication by the German Mathematical Society.

1. The Approximate Representation of a Function by Means of a Finite Trigonometric Series. Fourier's Constants. A finite series of the form:

$$S_k(x) = A + \sum_{n=1}^{n=k} (A_n \cos nx + B_n \sin nx), \tag{1}$$

where the A's and B's are supposed to be real constants, we will call a trigonometric series with $k+1$ terms.

It obviously represents a continuous periodic function of period 2π, which will depend on the values we assign to the coefficients A_n and B_n. Suppose now that we denote by $f(x)$ any real continuous periodic function of period 2π, and enquire: *how must the coefficients of the finite trigonometric series $S_k(x)$ be determined (the integer k being supposed given) so that it will give the best approximate representation of the function $f(x)$?* In order that this question may have a precise meaning we must agree on some criterion according to which we shall decide which of two given functions gives the better approximation to $f(x)$. Let us write

$$P_k(x) = f(x) - S_k(x).$$

We may say roughly that the smaller the numerical values of $P_k(x)$ the better is the approximation $S_k(x)$. It is however clear that a change in the coefficients may decrease the numerical value of $P_k(x)$ at some points while it increases it at other points. We will adopt, for the moment, the standpoint of

* Reference may be made to the first volumes of the treatises by Picard and Goursat. For a more elaborate discussion, which will be found very useful in clearing up the difficulties of the subject, see an article by Osgood, *Bull. Amer. Math. Soc.*, vol. 3, p. 59; 1896.

the method of least squares and say that the smaller the value of the integral

$$I_k = \int_{-\pi}^{\pi} \left\{ P_k(x) \right\}^2 dx,$$

the better is the approximation $S_k(x)$. Our problem then is to make this integral regarded as a function of the $2k+1$ coefficients ($A_0, A_1, \cdots A_k$; $B_1, \cdots B_k$) a minimum. For this purpose let us differentiate it partially with regard to the variables A_n and B_n. We have

$$(2) \qquad I_k = \int_{-\pi}^{\pi} \left\{ f(x) \right\}^2 dx - 2\int_{-\pi}^{\pi} f(x)\, S_k(x)\, dx + \int_{-\pi}^{\pi} \left\{ S_k(x) \right\}^2 dx\,;$$

accordingly, since $\dfrac{\partial S_k(x)}{\partial A_n} = \cos nx$ and $\dfrac{\partial S_k(x)}{\partial B_n} = \sin nx$,

$$\frac{\partial I_k}{\partial A_n} = -2\int_{-\pi}^{\pi} f(x)\cos nx\, dx + 2\int_{-\pi}^{\pi} S_k(x) \cos nx\, dx,$$

and

$$\frac{\partial I_k}{\partial B_n} = -2\int_{-\pi}^{\pi} f(x)\sin nx\, dx + 2\int_{-\pi}^{\pi} S_k(x) \sin nx\, dx.$$

If we replace $S_k(x)$ here by its value, each of the integrals in which it occurs breaks up into a sum of $2k+1$ simple integrals of one or the other of the following types, where p and q denote positive integers or zero:

$$(3)\left\{\begin{array}{ll}
\displaystyle\int_{-\pi}^{\pi} \cos px\, dx = \begin{cases} 0 & (p \neq 0), \\ 2\pi & (p = 0)\,; \end{cases} & \displaystyle\int_{-\pi}^{\pi} \sin px\, dx = 0\,; \\
\left.\begin{array}{l}
\displaystyle\int_{-\pi}^{\pi} \cos^2 px\, dx = \left[\frac{x}{2} + \frac{\sin 2px}{4p}\right]_{-\pi}^{\pi} = \pi \\
\displaystyle\int_{-\pi}^{\pi} \sin^2 px\, dx = \left[\frac{x}{2} - \frac{\sin 2px}{4p}\right]_{-\pi}^{\pi} = \pi
\end{array}\right\} & (p \neq 0)\,; \\
\multicolumn{2}{l}{\displaystyle\int_{-\pi}^{\pi} \sin px \cos qx\, dx = \tfrac{1}{2}\int_{-\pi}^{\pi}\Big[\sin(p+q)x + \sin(p-q)x\Big] dx = 0\,;} \\
\left.\begin{array}{l}
\displaystyle\int_{-\pi}^{\pi} \cos px \cos qx\, dx = \tfrac{1}{2}\int_{-\pi}^{\pi}\Big[\cos(p-q)x + \cos(p+q)x\Big] dx = 0 \\
\displaystyle\int_{-\pi}^{\pi} \sin px \sin qx\, dx = \tfrac{1}{2}\int_{-\pi}^{\pi}\Big[\cos(p-q)x - \cos(p+q)x\Big] dx = 0
\end{array}\right\} & (p \neq q).
\end{array}\right.$$

Using these values we find:

$$(4)\quad \begin{cases} \dfrac{\partial I_k}{\partial A_0} = -2\displaystyle\int_{-\pi}^{\pi} f(x)\,dx + 4\pi A_0, \\ \dfrac{\partial I_k}{\partial A_n} = -2\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx + 2\pi A_n \qquad (n \neq 0), \\ \dfrac{\partial I_k}{\partial B_n} = -2\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx + 2\pi B_n. \end{cases}$$

Equating these partial derivatives to zero gives us necessary conditions for a minimum which completely determine the values of the A's and B's. That this determination really makes I_k a minimum we see at once by a glance at the second derivatives of this quantity with regard to the A's and B's obtained by differentiating (4). The values of the coefficients thus determined we call with Hurwitz the *Fourier's Constants* of the function $f(x)$, except that for the sake of greater uniformity in the formulæ we take as the first of these constants not A_0 itself but $2A_0$. Our result may be stated as follows:

I. *If $f(x)$ is a periodic function of period 2π, real and continuous for all real values of x, and if k is a given integer positive or zero, the finite trigonometric series with $k+1$ terms*

$$(5)\qquad S_k(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{n=k}(a_n \cos nx + b_n \sin nx)$$

which gives the best approximate representation of this function in the sense of the theory of least squares, — i. e. which gives to the integral

$$(6)\qquad I_k = \int_{-\pi}^{\pi}\Big\{f(x) - S_k(x)\Big\}^2 dx$$

the smallest value, — is that in which the coefficients a_n and b_n are the Fourier's constants of $f(x)$:

$$(7)\qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.^*$$

* The application of the method of least squares to the determination of the coefficients of finite trigonometric series goes back to Bessel (cf. §10). The theorem here given may be regarded as a limiting case of Bessel's results, but was not explicitly stated by him. It was stated by Toepler, *Wiener Anzeigen,* vol. 13 (1876), p. 205.

Having thus determined the values of the coefficients which make the integral I_k a minimum, we can now easily determine the value of this minimum. The last two integrals which occur in formula (2) may be at once evaluated by means of formulæ (7) and (3). We thus find:

$$\int_{-\pi}^{\pi} f(x)\, S_k(x)\, dx = \int_{-\pi}^{\pi} \left\{ S_k(x) \right\}^2 dx = \pi \left[\frac{a_0^2}{2} + \sum_{n=1}^{n=k} (a_n^2 + b_n^2) \right].$$

Substituting these values in (2) we have the result:

II. *The minimum value of the integral I_k is given by the formula*

$$(8) \qquad \overline{I_k} = \int_{-\pi}^{\pi} \left\{ f(x) \right\}^2 dx - \pi \left[\frac{a_0^2}{2} + \sum_{n=1}^{n=k} (a_n^2 + b_n^2) \right].*$$

By taking a larger and larger number of terms in our series (5) we shall get better and better approximations to our function $f(x)$, as is seen either directly or by a glance at formula (8). The question thus naturally suggests itself whether the infinite trigonometric series

$$(9) \qquad \frac{a_0}{2} + \sum_{n=1}^{n=\infty} (a_n \cos nx + b_n \sin nx),$$

in which the coefficients are determined as the Fourier's constants of $f(x)$, may not give us, no longer an approximate, but a perfect representation of $f(x)$. This is the fundamental question which lies at the foundation of the whole theory of Fourier's series. Any series of the form (9), whatever its coefficients may be, we will call a trigonometric series † while it is only when the coefficients of (9) are determined from some function $f(x)$ by formulæ (7) that we speak of (9) as a *Fourier's Series*, or more explicitly as *the* Fourier's development of $f(x)$. In using these terms we do not intend in any way to prejudice the question as to whether the series thus formed really represents the function

* We note in passing that we find by exactly the same method that the best approximation (understanding this term in the sense above adopted) to $f(x)$ by a function of the form $\sum_{k_1}^{k_2}(a_n \cos nx + b_n \sin nx)$, where k_1 and k_2 are given positive integers, is obtained by taking for the coefficients a_n and b_n the Fourier's constants of $f(x)$; and that if we denote by $P(x)$ the difference between this function and $f(x)$ we have

$$\int_{-\pi}^{\pi} \left\{ P(x) \right\}^2 dx = \int_{-\pi}^{\pi} \left\{ f(x) \right\}^2 dx - \pi \sum_{k_1}^{k_2} (a_n^2 + b_n^2).$$

† There are of course other forms of trigonometric series, but, since no others are to be considered in this article, no ambiguity will arise.

$f(x)$ or even whether it converges. We shall however ultimately find that whenever the Fourier's development converges it does represent the function $f(x)$, and that, although there are cases where the series will diverge, these cases will not present themselves in any of the large classes of functions to which one ordinarily wishes to apply this method of development.

We shall not attempt to use the method of least squares, which has led us naturally to the consideration of Fourier's series, to treat the question of the convergence of these series.* There is, however, one inference which may be immediately drawn from formula (8) and which will prove useful to us later. Since by (6) $\overline{I_k}$ cannot be negative, formula (8) shows us that

$$\frac{a_0^2}{2} + \sum_{n=1}^{n=k} (a_n^2 + b_n^2) \leqq \frac{1}{\pi} \int_{-\pi}^{\pi} \left\{ f(x) \right\}^2 dx.$$

Accordingly if we allow k to become infinite, the series we obtain on the left must converge since its terms are all positive and the sum of its first $n + 1$ terms does not become infinite with n.† Hence the following theorem:

III. *If a_n and b_n are the Fourier's constants of any continuous periodic function $f(x)$, both the series*

$$\sum_1^\infty a_n^2, \qquad \sum_1^\infty b_n^2$$

will converge, and therefore

$$\lim_{n=\infty} a_n = \lim_{n=\infty} b_n = 0.$$

The foregoing results may be generalized if we note that the requirement we have so far made that $f(x)$ be continuous is not necessary. This function may be allowed to have a finite number of discontinuities in the interval $-\pi \leqq x \leqq \pi$ provided it remains finite at these points, neither the statements nor the proofs of the theorems we have established being in any way modified

* An interesting attempt to do this was made by Harnack, *Math. Ann.*, vol. 17 (1880), p. 123. This paper however must be used with great caution.

† If we could show that I_k approaches zero as k become infinite we should have established the formula

$$\frac{a_0^2}{2} + \sum_1^\infty (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} \{ f(x) \}^2 dx.$$

This formula, which Harnack attempted to prove, has recently been established by other methods; cf. §4, formula (25), of the present paper. The theorem of the text and its proof are due to Harnack.

by this generalization. In fact it is not necessary to require even that $f(x)$ be finite at the points of discontinuity, but merely that $\int \{ f(x) \}^2 dx$ converge when extended over any finite interval. This last generalization, however, being of less importance to us, we note merely the following results:

IV. *Theorems* I, II, III *still hold if the requirement that $f(x)$ be everywhere continuous be replaced by the requirement that it be finite and have in the interval* $-\pi \leqq x \leqq \pi$ *only a finite number of discontinuities.**

We now state for future reference the following facts concerning Fourier's constants which follow immediately from their definition (7).†

V. *If* $F(x) = f_1(x) \pm f_2(x)$, *each Fourier's constant of* F *is the sum or difference of the corresponding Fourier's constants of* f_1 *and* f_2.

VI. *If* $F(x) = cf(x)$, *where* c *is a constant, each Fourier's constant of* F *is* c *times the corresponding Fourier's constant of* f.

VII. *If* $F(x) = f(x - a)$ *and if* a_n *and* b_n *are the Fourier's constants of* $f(x)$, *then the Fourier's constants of* $F(x)$ *will be*

$$A_n = a_n \cos na - b_n \sin na,$$
$$B_n = a_n \sin na + b_n \cos na.$$

If in this last theorem (and a similar remark applies to the first two) we were to substitute in the Fourier's development of $f(x)$ the quantity $x - a$ in place of x, we should obtain a new trigonometric series whose coefficients are precisely the quantities A_n, B_n last written. This, however, would not constitute a proof of VII for two reasons. First because we know nothing about the convergence of the Fourier's expansion of $f(x)$, and secondly because even if we did know that this expansion really converges and represents $f(x)$ we should have no reason to feel sure that the expansion for $F(x)$ obtained by the substitution is its Fourier's expansion, and not some other trigonometric expansion of this function.

Finally we come to the following theorem which is one of the oldest general theorems in the whole theory of trigonometric series, going back as it does well into the eighteenth century:‡

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* Or more generally that it be finite and integrable in Riemann's sense.

† In all of these theorems we will assume, in order to ensure the existence of the Fourier's constants, that $\int | f(x) | dx$, or in V $\int | f_1(x) | dx$, converges when extended over any finite interval.

‡ It was explicitly stated by Euler in a paper written in 1777. Cf. on this subject Burkhardt's *Bericht*, p. 70.

VIII. *If a trigonometric series converges in such a way that after it is multiplied by any continuous function we have a right to integrate it term by term, then this series is the Fourier's development of the function which it represents.*

It should be noticed that the conditions of this theorem are fulfilled if the trigonometric series converges uniformly for all values of x, and also in many cases of non-uniform convergence.

To prove this theorem we write the series in the form (9) and call its value $f(x)$. If we multiply first by $\cos nx$ and then by $\sin nx$ and integrate each time from $-\pi$ to π we get, by using formulæ (3),

$$\int_{-\pi}^{\pi} f(x) \cos nx \, dx = \pi a_n, \quad \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \pi b_n.$$

Thus we see that a_n and b_n are the Fourier's constants of $f(x)$ and the theorem is proved.

2. Introduction of Convergence Factors. Since no complications will be thereby introduced we will consider in this section functions of a somewhat more general character even than those referred to in theorem IV of the last section. We will denote the independent variable by ϕ and will let $f(\phi)$ be any real function with period 2π * which in the interval $-\pi \leqq \phi \leqq \pi$ has at most a finite number of discontinuities, these discontinuities being of such a nature that $\int |f(\phi)| d\phi$ converges when extended over any portion of this interval. We thus admit the following kinds of discontinuity:

(*a*) *Finite jumps.* The function $f(\phi)$ is said to have a finite jump of magnitude D at the point ϕ_0 if it is discontinuous at ϕ_0 but approaches a finite limit which we will call $f(\phi_0 + 0)$ when ϕ approaches ϕ_0 from above, and a finite limit which we will call $f(\phi_0 - 0)$ when ϕ approaches ϕ_0 from below, and if $f(\phi_0 + 0) - f(\phi_0 - 0) = D$. It should be understood that *at* the point ϕ_0 the function need not be defined at all, and if defined need not have either of the two values $f(\phi_0 + 0)$ or $f(\phi_0 - 0)$.† In fact the notation here used, which is essentially that of Dirichlet, though convenient is somewhat misleading since

* A function is said to have the period $2p$ if, ϕ_0 being any constant, the function is either not defined at either of the points ϕ_0 and $\phi_0 + 2p$, or is defined at both of them, and has the same value at both.

† It of course cannot have both, since we suppose our function to be single valued wherever defined.

the quantities $f(\phi \pm 0)$ are not the values which the function takes on at specified points, but merely the limits approached by such values.*

(*b*) *Finite discontinuities.* The point ϕ_0 is said to be a finite discontinuity of f if there exists a constant M such that throughout the neighborhood of ϕ_0 we have $|f(\phi)| < M$. These finite discontinuities include the finite jumps mentioned above as a special case, but they also include such discontinuities as the function $\sin(1/\phi)$ has at $\phi = 0$.

(*c*) *Certain discontinuities where $f(\phi)$ does not remain finite*, such for instance as the following functions have at the point $\phi = \phi_0$:

$$\frac{1}{\sqrt{\phi - \phi_0}}, \quad \frac{\sin[1/(\phi - \phi_0)]}{\sqrt{\phi - \phi_0}}, \quad \sqrt{\operatorname{ctn}(\phi - \phi_0)}\,;$$

or more generally any function of the form

$$f(\phi) = (\phi - \phi_0)^{a} F(\phi),$$

where $0 > a > -1$, and $F(\phi)$ remains finite at ϕ_0.

For functions of this sort the integrals which appear in the Fourier's constants

$$(10) \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\cos n\phi\, d\phi, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\sin n\phi\, d\phi$$

evidently converge, and, if we introduce the positive constant

$$M = \frac{1}{\pi}\int_{-\pi}^{\pi} |f(\phi)|\, d\phi,$$

we have :†

$$(11) \qquad |a_n| \leqq M, \qquad |b_n| \leqq M \qquad (n = 0, 1, 2, \cdots).$$

* The case of a finite jump of magnitude zero deserves special mention. Here the function approaches a definite limit $f(\phi_0 + 0) = f(\phi_0 - 0)$ as ϕ approaches ϕ_0 in any way. If $f(\phi_0)$ were defined as this limiting value we should have no discontinuity at all. The discontinuity is due either to a total lack of definition at ϕ_0, or to the fact that $f(\phi_0) \neq f(\phi_0 \pm 0)$. This is Riemann's "hebbare Unstetigkeit." Cf. Pierpont, *Functions of Real Variables*, vol. I, p. 212.

† On account of the well known theorem: if $\int_a^b |f(\phi)|\, d\phi$ converges, and $|F(\phi)| \leqq |f(\phi)|$ then $\int_a^b F(\phi)\, d\phi$ converges, and $|\int_a^b F(\phi)d\phi| \leqq \int_a^b |f(\phi)|\, d\phi$.

These inequalities are far from being enough to enable us to establish the convergence of the Fourier's series

$$(12)\qquad \frac{a_0}{2} + \sum_1^\infty (a_n \cos n\phi + b_n \sin n\phi).$$

We therefore follow Poisson* and introduce *Convergence Factors* r^n into this series thus forming the new series

$$(13)\qquad \frac{a_0}{2} + \sum_1^\infty r^n(a_n \cos n\phi + b_n \sin n\phi).$$

If we interpret (r, ϕ) as polar coordinates and confine our attention for the moment to points within and on the circumference of the circle $r = R$, where the positive constant R is less than 1, we see from the inequalities (11) that no term of the series (13) can exceed in numerical value the corresponding term of the series

$$\frac{1}{2} M + \sum_1^\infty 2MR^n;$$

and, this being a convergent series of positive constant terms, it follows that (13) is absolutely and uniformly convergent when $r \leqq R$. Accordingly, since the terms of (13) are continuous functions of (r, ϕ), we have the theorem:

I. *The series* (13) *converges absolutely when* $r < 1$. *It converges uniformly when* $r \leqq R$, *where* R *is a positive constant less than* 1. *It represents a function* $F(r, \phi)$ *which is a continuous function of* (r, ϕ) *when* $r < 1$.

This theorem, however, gives us no information either as to the convergence of (13) when $r = 1$, or as to whether the function $F(r, \phi)$ approaches a limit as r approaches 1. This last point we proceed, with Poisson, to settle by actually summing the series (13).

By introducing the values (10) of the coefficients into (13) we get:

$$(14)\ F(r, \phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\psi)d\psi + \frac{1}{\pi}\sum_1^\infty \int_{-\pi}^{\pi} r^n f(\psi)\cos n(\psi - \phi)d\psi \quad (r<1).$$

Now consider the series

$$(15)\qquad 1 + r\cos\theta + r^2\cos 2\theta + \cdots \qquad (r < 1).$$

* *Journal de l'École Polytechnique*, Cahier 18 (1820), p. 422. Poisson uses $t = -\log r$, so that his convergence factors have the form e^{-nt}. Cf. also Sommerfeld's dissertation: *Die willkürlichen Functionen in der mathematischen Physik*, Königsberg, 1891.

This is the real part of the geometric series

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots \qquad (|z| < 1),$$

where $z = r(\cos\theta + i\sin\theta)$. Accordingly (15) converges to the value

$$\frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2}.$$

Replacing θ by $\psi - \phi$ we get, after subtracting $\frac{1}{2}$ from the series (15),

$$\text{(16)} \qquad \frac{\frac{1}{2}(1-r^2)}{1 - 2r\cos(\psi - \phi) + r^2} = \frac{1}{2} + \sum_1^\infty r^n \cos n(\psi - \phi) \qquad (r < 1).$$

If we regard r, ϕ as constants, this series is obviously uniformly convergent for all values of ψ. Multiplying it through by $f(\psi)/\pi$ and integrating term by term with regard to ψ from $-\pi$ to π gives us precisely the series (14).* Accordingly:

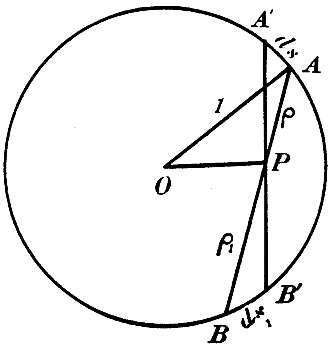
$$\text{(17)} \qquad F(r, \phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\psi)\,\frac{1 - r^2}{1 - 2r\cos(\psi - \phi) + r^2}\,d\psi \qquad (r < 1).$$

This is *Poisson's Integral* to whose discussion we will devote the next section. In doing this we follow out in modified form the line of thought distinctly marked out by Poisson and carried through in rigorous form by Schwarz.†

* We are here using the theorem: *If $u_1(x) + u_2(x) + \cdots$ is throughout the interval $a \leqq x \leqq b$ a uniformly convergent series of continuous (or more generally of finite and integrable) functions, and $f(x)$ is a function such that $\int_a^b |f(x)|\,dx$ is convergent, then the series $\Sigma\, f(x)u_n(x)$ may be integrated from a to b term by term.* It is true that this last written series is not in general uniformly convergent if $f(x)$ does not remain finite, but the same kind of reasoning that is used to show that a uniformly convergent series of continuous functions may be integrated term by term may be readily applied here.

† See *Crelle*, vol. 74 (1872), p. 218. The treatment we shall give depends in part on a remark of Schwarz, *Coll. Works*, vol. 2, p. 360, partly on a note by the writer, *Bull. Amer. Math. Soc.*, vol. 4 (1898), p. 424.

3. The Theory of Poisson's Integral. Let O be the origin, P the point whose polar coordinates are (r, ϕ), A the point on the unit circle with coordinates $(1, \psi)$, and B the point where the line AP meets the circle again. This latter point we will speak of as being *opposite to A with regard to P.* Let $PA = \rho$, $PB = \rho_1$, and let us denote by s the length of the arc of the circle from the point where $\psi = 0$ to the point A, so that $s = \psi$.



The denominator of the fraction in (17) is ρ^2, its numerator which is $(1 - r)(1 + r)$ may, by a familiar geometrical theorem, be written $\rho\rho_1$. Thus Poisson's Integral takes the form :

$$(18) \qquad F(r, \phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\,\frac{\rho_1}{\rho}\,ds.$$

Now consider a second chord through P which meets the circle in a point A' near to A and in a point B' near to B. Call the arcs AA' and BB' ds and ds_1 respectively. By similar triangles we have $PA/PB' = AA'/BB'$, and if we regard ds as an infinitesimal, these ratios differ by infinitesimals from ρ/ρ_1 and ds/ds_1 respectively. Hence Poisson's Integral may be written :

$$(19) \qquad F(r, \phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\,ds_1.$$

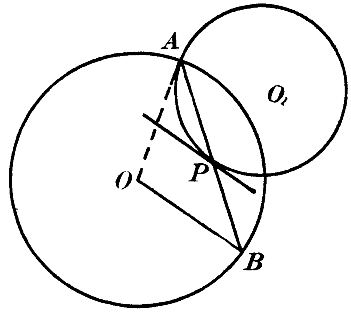
This very simple form is due to Schwarz and was interpreted by him in the following manner :

I. *If the value which the function $f(s)$ has at every point of the unit circle be transferred to the opposite point with regard to P, the arithmetic mean of the resulting distribution of values is equal to the value of F at P.*

Another interpretation which avoids the necessity of any shifting of the values of $f(s)$ may be reached as follows.

We note first the simple lemma :

If two circles with centres at O and O_1 intersect orthogonally at A, and if a chord through A cuts the circles at B and P respectively, then the radius OB of the first circle is parallel to the tangent drawn at P to the second.

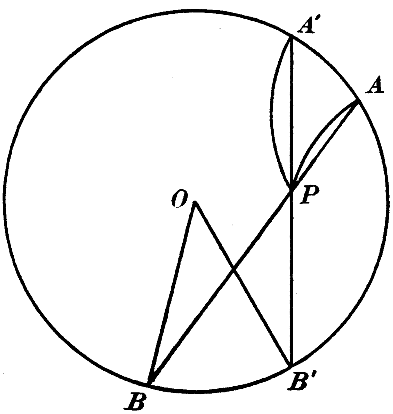


For the radius OA of the first circle is tangent to the second, and therefore the angle which the tangent at P makes with AB is equal to the angle OAB which in turn is equal to OBA.

We may state this lemma a little more precisely, if, as is the case in our figure, P lies within the first circle. Then the direction of the radius OB is exactly *opposite* to the direction at P of the arc PA, this arc being understood to mean that part of the second circle bounded by the two points P and A which lies wholly within the first circle.

The following theorem now follows at once.

II. *Given a circle C with centre at O, two points A and A' upon it, and a point P within it. Two arcs of circles PA and PA' are drawn within C cutting C orthogonally. If the straight lines PA and PA' meet C again in B and B' respectively, then the angle BOB' is equal both in magnitude and in direction to the angle between the* ARCS APA'.



For the directions at P of the arcs PA and PA' are by our lemma exactly opposite to the directions of the radii OB and OB'.

Let us now, returning to Poisson's Integral, denote by θ the angle which a circle through P cutting the unit circle orthogonally at the variable point A makes with a fixed circle of the same sort. Then by the theorem just proved

$$d\theta = ds_1,$$

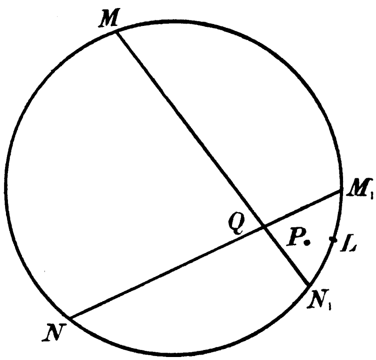
and accordingly Poisson's Integral may be written:

$$F(r,\phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\,d\theta. \tag{20}$$

This formula may be interpreted as follows:

III. *If we imagine that at each point on the unit circle the value of $f(s)$ at that point has been marked, then the value of $F(r, \phi)$ at any point P within the circle is equal to the average of these values as they would be read off by an observer at P who turns with uniform angular velocity and who is situated in a refracting medium which causes the rays of light reaching his eye to take the form of circular arcs orthogonal to the unit circle.*

Let us now consider the angle formed at P by two arcs through P orthogonal to the given circle and meeting it in M and N. There are of course two such angles whose sum is 2π. These we may regard as subtended respectively by the two arcs of the circle bounded by M and N. Let us choose one of these two arcs and call it the arc MN, and consider the angle between the two circular arcs at P which is subtended by MN. If now, M and N remaining fixed, P approaches as its limiting position a point L on the given circle not on the arc MN, the angle at P evidently changes continuously, and since it is zero when P coincides with L it must be approaching zero as its limit. Hence the lemma:



IV. *If L, M, N are three distinct points on a circle, and η any positive constant, a region can be marked off about the point L such that if P is any point within this region and also within the circle, and if two circular arcs PM and PN are drawn within the given circle and orthogonal to it, the angle between these arcs subtended by the arc MN (not MLN) is less than η.*

A proof of this lemma a little more complete than that given above is the following. For the sake of convenience of statement take the radius of the circle as unit of length. Lay off from L two arcs of length $\eta/2$, namely LM_1 in the direction LM, and LN_1 in the direction LN. Draw the chords MN_1 and NM_1 intersecting in Q, a point in the circle. Then the region bounded by the two straight lines QM_1 and QN_1 and by the arc M_1LN_1 is a

region of the kind required.* For the angle between the arcs PM and PN is measured, according to II, by the arc intercepted between the points where the lines MP and NP meet the circle again; and since these points must lie between M_1 and N_1, the length of this arc is less than η.

We are now in a position to consider the important question whether the function $F(r, \phi)$ approaches a limit as the point (r, ϕ) approaches a point $(1, \phi_0)$ on the circumference of the unit circle. In doing this we will at first suppose that $f(\psi)$ is continuous when $\psi = \phi_0$. Let us denote as before by P the point (r, ϕ), by L the point $(1, \phi_0)$. Interpreting the value of F at P by means of III, we see from IV that by bringing P near enough to L the angle which any arc of the unit circle on which L does not lie appears to the observer at P to subtend can be made as small as we please, and therefore the values which $f(\psi)$ has on this arc will have less and less influence on the value of the average F. The only points which, when P comes to lie very near to L, have any appreciable influence on this average are the points in the immediate vicinity of L, and therefore this average comes nearer and nearer to the value of $f(\psi)$ at L. Hence the following important result discovered by Poisson:

V. *If $f(\psi)$ is continuous at the point $\psi = \phi_0$, then*

$$\lim_{\substack{r=1\\ \phi=\phi_0}} F(r, \phi) = f(\phi_0),$$

it being understood that in this limiting process r and ϕ are independent variables, and $r < 1$.

The formal proof of this theorem, of which the reasoning just given may be regarded as a rough sketch, is as follows.†

Consider first the function

$$F_1(r, \phi) = \int_{\theta_1}^{\theta_2} f(s)\,d\theta,$$

and suppose that L does not lie on the arc $\theta_1 \leqq \theta \leqq \theta_2$. We wish to prove that this function approaches zero as P approaches L; that is, that however small the positive quantity ϵ may be, we can surround L by a region so small that for all points P within it and also within the unit circle $|F_1(r, \phi)| < \epsilon$.

* A much larger region could easily be found. The locus of P when the angle in question is equal to η is readily shown to be the arc NQM of the circle through these three points, and the largest region throughout which this angle is less than η is the crescent bounded by the arc just mentioned and the arc NLM.

† This proof, which covers the next two and a half pages, may be omitted if the reader wishes to do so.

If f remains finite on the arc $\theta_1 \leqq \theta \leqq \theta_2$, denote by M a positive quantity such that for all points on this arc $| f(s) | < M$. Then

$$| F_1(r, \phi) | < \int_{\theta_1}^{\theta_2} M d\theta = M(\theta_2 - \theta_1) ;$$

and now take a region around L so small that for all points within it and also within the unit circle

$$\theta_2 - \theta_1 < \frac{\epsilon}{M} \cdot$$

The possibility of doing this is established by IV. For points within this region we have then $|F(r, \phi)| < \epsilon$ as was to be shown.

We must, however, still consider the case in which at one or more points on the arc $\theta_1 \leqq \theta \leqq \theta_2$ the function $f(s)$ fails to remain finite. For this purpose let us determine the position of points on this arc by means of the angle ψ measured from the centre of the circle; and denote by ψ_1 and ψ_2 the values of ψ corresponding to the extremities θ_1 and θ_2 of this arc. Denote the points in whose neighborhood f fails to remain finite on this arc by $\psi = a_1, a_2, \cdots a_k$, and mark off a neighborhood around each of these points so small that $\int |f(\psi)| d\psi$ extended over any one of these neighborhoods is less than $\epsilon/2k$. Let us now break up the function $f(\psi)$ into two parts:

$$f(\psi) = f_1(\psi) + f_2(\psi),$$

where $f_1(\psi)$ is equal to zero everywhere except in the neighborhoods of $a_1, \cdots a_k$ just determined, while in these neighborhoods it is equal to $f(\psi)$; accordingly $\int_{\psi_1}^{\psi_2} |f_1(\psi)| d\psi < \frac{\epsilon}{2}$. On the other hand $f_2(\psi)$ will be zero in the neighborhoods of $a_1, \cdots a_k$ and will be equal to $f(\psi)$ everywhere else. It will therefore be finite throughout the whole arc, so that from what was just proved a region can be marked off about L such that when P is in this region:

$$\left| \int_{\theta_1}^{\theta_2} f_2(\psi) d\theta \right| < \frac{\epsilon}{2} \cdot$$

Now we have:

$$F_1(r, \phi) = \int_{\theta_1}^{\theta_2} f_1(\psi) d\theta + \int_{\theta_1}^{\theta_2} f_2(\psi) d\theta,$$

and accordingly it merely remains to show that a region can be marked off about L such that, when P is within it, the first of these two integrals is in absolute value less than $\epsilon/2$. The reasoning which took us above from formula (18) to formula (20) shows at once that:

$$\int_{\theta_1}^{\theta_2} f_1(\psi)\,d\theta = \int_{\psi_1}^{\psi_2} f_1(\psi)\,\frac{\rho_1}{\rho}\,d\psi,$$

and now it is easy to see that a region can be marked off around L such that when P lies within it $\rho_1 < \rho$ for all points ψ on the arc $\psi_1 \leqq \psi \leqq \psi_2$.* Accordingly for points P in this region

$$\left|\int_{\theta_1}^{\theta_2} f_1(\psi)\,d\theta\right| \leqq \int_{\psi_1}^{\psi_2} |f_1(\psi)|\,d\psi < \frac{\epsilon}{2}\cdot$$

Thus we have completed the proof that in all cases $F_1(r, \phi)$ approaches zero as P approaches L.

Let us now come back to the function $F(r, \phi)$ itself. We may write:

$$F(r, \phi) - f(\phi_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big[f(\psi) - f(\phi_0)\Big]d\theta.$$

Let us now determine a positive δ such that when $|\psi - \phi_0| < \delta$, $|f(\psi) - f(\phi_0)| < \epsilon/2$. This is possible since we have assumed f to be continuous at ϕ_0. For the sake of simplicity of notation we will measure the angle θ so that when $\psi = \phi_0$, $\theta = 0$, and we will call the value of θ when $\psi = \phi_0 + \delta$, ξ, and when $\psi = \phi_0 - \delta$, $-\xi$. We have:

$$F(r, \phi) - f(\phi_0) = \frac{1}{2\pi}\int_{-\xi}^{\xi}\Big[f(\psi) - f(\phi_0)\Big]d\theta + \frac{1}{2\pi}\int_{\xi}^{2\pi-\xi}\Big[f(\psi) - f(\phi_0)\Big]\,d\theta.$$

The second of these integrals approaches zero as its limit as P approaches L since it is of the form of the integral F_1 considered above, except that $f(\psi)$ is replaced by $f(\psi) - f(\phi_0)$. Accordingly by restricting P to a certain neighborhood of L we have:

$$\frac{1}{2\pi}\left|\int_{\xi}^{2\pi-\xi}\Big[f(\psi) - f(\phi_0)\Big]d\theta\right| < \frac{\epsilon}{2}\cdot$$

* For when P lies at L, $\rho_1 = 0$ and ρ is for all values of ψ on the arc at least as great as the distance from P to the nearer end of the arc; and as P moves away from L both ρ and and ρ_1 vary continuously. It is easy to see that the *largest* region of this sort about L is the region bounded by the arc of the circle on which L lies and by two circles touching the unit circle at ψ_1 and ψ_2 respectively and passing through the origin.

We also have:

$$\frac{1}{2\pi}\left|\int_{-\xi}^{\xi}\Big[f(\psi)-f(\phi_0)\Big]d\theta\right| \leqq \frac{1}{2\pi}\int_{-\xi}^{\xi}|f(\psi)-f(\phi_0)|\,d\theta < \frac{\epsilon}{2}\cdot\frac{2\xi}{2\pi} < \frac{\epsilon}{2}.$$

Accordingly

$$|F(r,\phi)-f(\phi_0)| < \epsilon,$$

and our theorem is proved.

We might now go on to the consideration of the behavior of $F(r, \phi)$ as P approaches a point L on the unit circle at which f is not continuous. Although various theorems might be established here,* we will prove only one which we shall find useful later:

VI. *If $f(\psi)$ has a finite jump at ϕ_0,*

$$\lim_{r=1} F(r,\phi_0) = \tfrac{1}{2}\left[f(\phi_0+0)+f(\phi_0-0)\right].$$

It should be noticed that this is a much more special theorem than the last since (r, ϕ) is now approaching its limiting position $(1, \phi_0)$ along a radius. Let us again measure θ so that when $\psi = \phi_0$, $\theta = 0$. We will break up Poisson's integral into two parts, thus:

$$F(r,\phi) = \frac{1}{2\pi}\int_0^{\pi} f(\psi)\,d\theta + \frac{1}{2\pi}\int_{-\pi}^{0} f(\psi)\,d\theta,$$

and prove that the first part approaches $\tfrac{1}{2}f(\phi_0+0)$ as its limit, the second $\tfrac{1}{2}f(\phi_0-0)$. That this is so is very evident from the interpretation III of

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* For instance the following:

If $f(\psi)$ becomes positively (negatively) infinite at ϕ_0, $F(r, \phi)$ becomes positively (negatively) infinite as (r, ϕ) approaches $(1, \phi_0)$ in any way.

If we denote by M and m the upper and lower limits of indetermination (cf Stolz, *Theoretische Arithmetic,* p. 169) *of f at the point ϕ_0, then, however small the positive quantity ϵ, a neighborhood of $(1, \phi_0)$ can be marked off throughout which*

$$m-\epsilon < F(r,\phi) < M+\epsilon.$$

A special case of this last, is the following:

If $f(\psi)$ has a finite jump at ϕ_0, then, no matter how small the positive quantity ϵ, a neighborhood of $(1, \phi_0)$ can be marked off in which

$$f(\phi_0-0)-\epsilon < F(r,\phi) < f(\phi_0+0)+\epsilon \qquad \textit{if} \quad f(\phi_0-0) < f(\phi_0+0),$$
$$f(\phi_0+0)-\epsilon < F(r,\phi) < f(\phi_0-0)+\epsilon \qquad \textit{if} \quad f(\phi_0+0) < f(\phi_0-0).$$

The truth of all these theorems is evident from the interpretation III of Poisson's integral, and the formal proof is easy.

Poisson's integral. The formal proof, which we give only for the first part, since it is precisely similar for the second, is as follows :

The positive quantity ϵ being given at pleasure, let us determine a positive δ such that when $\phi_0 < \psi < \phi_0 + \delta$, $|f(\psi) - f(\phi_0 + 0)| < \epsilon/2$, and when $\psi = \phi_0 + \delta$ let $\theta = \zeta$. Then

$$\frac{1}{2\pi}\int_0^\pi f(\psi)d\theta - \tfrac{1}{2}f(\phi_0+0) = \frac{1}{2\pi}\int_0^\zeta [f(\psi) - f(\phi_0+0)]d\theta + \frac{1}{2\pi}\int_\zeta^\pi [f(\psi) - f(\phi_0+0)]d\theta.$$

The second of these expressions, by what we proved above about the function we called F_1, approaches zero as r approaches 1. The first is in absolute value less than $\frac{\epsilon}{2}\frac{\zeta}{2\pi} < \frac{\epsilon}{2}$. Accordingly for all values of r sufficiently near to 1 we have

$$\left|\frac{1}{2\pi}\int_0^\pi f(\psi)d\theta - \tfrac{1}{2}f(\phi_0+0)\right| < \epsilon,$$

and the proof that the first half of the above expression for F approaches $\frac{1}{2}f(\phi_0 + 0)$ is complete.

4. Applications of the Theory of Poisson's Integral. The results of the last section show that if $f(\phi)$ is any real function with period 2π which in the interval $-\pi \leqq \phi \leqq \pi$ has only a finite number of discontinuities and is such that $\int |f(\phi)| d\phi$ converges when extended over any portion of this interval, then f may be obtained at any point where it is continuous as the *limit* of an infinite trigonometric series as the coefficients of this series approach, in a certain manner, the Fourier's constants of f. Since the formation of an infinite series is itself a limiting process, we have to deal here with a double limit and our result may be written :

$$(21)\qquad f(\phi) = \lim_{r=1} \lim_{k=\infty} \left[\frac{a_0}{2} + \sum_1^k (a_n \cos n\phi + b_n \sin n\phi) r^n\right] \qquad (r > 1),$$

where the a's and the b's are the Fourier's constants of f. If we had the right to reverse the order in which these limits are taken we should have the result that f is represented by its Fourier's development. *But we do not have this right;* and there are actually cases in which f, though continuous, is not represented by its Fourier's development.

Although this most obvious application of the theory of Poisson's integral is impossible, there are certain other important applications to the theory of Fourier's series on the one hand and to the theory of finite trigonometric series on the other, which we will now develop.

Suppose that ϕ_0 is a point at which f is continuous. Then the function

$$F(r, \phi_0) = \frac{a_0}{2} + \sum_1^{\infty} (a_n \cos n\phi_0 + b_n \sin n\phi_0) r^n \qquad (r < 1)$$

will, by theorem V of the last section, approach $f(\phi_0)$ as its limit as r approaches 1. But by a well known theorem of Abel * we know that if a power series in r, such as this, converges when $r = 1$ to the value A, the function represented by the series when $r < 1$ will approach the value A as r approaches 1. Thus we have established the important theorem :†

I. *If the real function $f(\phi)$ with the period 2π has in the interval $-\pi \leqq \phi \leqq \pi$ only a finite number of discontinuities and is such that $\int |f(\phi)| d\phi$ converges when extended over any part of this interval, then if the Fourier's development of $f(\phi)$ converges at a point ϕ_0 where f is continuous, it will converge to the value $f(\phi_0)$.*

A second theorem of the same sort, whose proof follows in the same way from theorem VI of the last section is this :

II. *If $f(\phi)$ satisfies the same conditions that were imposed in* I *and if ϕ_0 is a point at which $f(\phi)$ makes a finite jump, then if the Fourier's development of $f(\phi)$ converges at ϕ_0, it will converge to the value*

$$\tfrac{1}{2}[f(\phi_0 + 0) + f(\phi_0 - 0)]. ‡$$

* For a proof see, for instance, Picard's treatise, vol. 1, p. 220.

†See Bonnet, *Mémoires de l'Académie de Belgique*, vol. 23 (1850), p. 11. This is the only application of Poisson's integral of which we shall make essential use in the subsequent sections. The rest of this section may be omitted on a first reading if so desired.

‡ The following further theorems may be obtained in the same way from the theorems of the foot-note on p. 98. In all of them we suppose f to satisfy the conditions stated in I.

If $f(\phi)$ becomes positively (or negatively) infinite at ϕ_0, its Fourier's development will not converge at that point.

If $f(\phi)$ has a finite discontinuity at ϕ_0, its Fourier's development, if it converges at ϕ_0, will converge there to a value lying between the limits of indetermination at that point; and in particular, if m_+ and M_+ are the lower and upper limits of indetermination for $\phi_0 + 0$, and m_-, M_- the similar limits for $\phi_0 - 0$, the value of the Fourier's development at ϕ_0, if it converges there, will lie between $\frac{1}{2}(m_+ + m_-)$ and $\frac{1}{2}(M_+ + M_-)$.

Further results of the same sort may be obtained by using the extension of Abel's theorem which says (cf. Stolz: *Allgemeine Arithmetik*, vol. 1, p. 279) that if a power series in r

An immediate result from I is the theorem:

III. *If all the Fourier's constants of a real, continuous function with period 2π are zero, the function is identically zero.**

In fact it is not necessary to require that the function be continuous, but merely that it satisfy the conditions stated in I. In this case, however, from the vanishing of all its Fourier's constants we can merely infer that the function must vanish at all points where it is continuous. From this generalized form of the theorem we infer at once the further result:

IV. *If two functions $f_1(\phi)$ and $f_2(\phi)$ have the period 2π and each has in the interval $-\pi \leqq x \leqq \pi$ at most a finite number of discontinuities, and the integrals $\int |f_1(\phi)|\,d\phi$, $\int |f_2(\phi)|\,d\phi$ converge when extended over any part of this interval, then if all the Fourier's constants of f_1 are respectively equal to the corresponding Fourier's constants of f_2, f_1 and f_2 will be equal except perhaps at the points where one or both of them are discontinuous.*

For $f_1(\phi) - f_2(\phi)$ is a function which satisfies the conditions in I and whose Fourier's constants are all zero.

We proceed now to another class of facts of which the simplest is the following:

V. *If $f(\phi)$ is a continuous function with the period 2π, and if into the Fourier's development of $f(\phi)$ we introduce the convergence factors r^n $(r < 1)$, the resulting series, which always converges, approaches* UNIFORMLY *the value $f(\phi)$ as r approaches 1.*

To prove this let us define a function $F(r, \phi)$ for all points within and on the circle $r = 1$, the definition for points within this circle being the same as the definition of F in §2 (i. e. the value of the Fourier's series

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diverges to $+\infty$ (or to $-\infty$) when $r = 1$ and converges when $|r| < 1$, then the function represented by the series becomes positively (or negatively) infinite as r approaches 1. From this we infer the theorem:

The Fourier's development of $f(\phi)$ cannot diverge to $+\infty$ (or to $-\infty$) except at points where $f(\phi)$ is positively (or negatively) infinite.

* This is practically a very special case of a theorem enunciated, though not satisfactorily proved, by Liouville in the first volume of his journal (1836) p. 261 (cf. §10 of the present paper). The theorem follows also at once from formula (25) below:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(\phi)]^2\,d\phi = \frac{a_0^2}{2} + \sum_1^{\infty}(a_n^2 + b_n^2).$$

Kneser has recently shown (*Sitzungsberichte d. Berl. math. Ges.*, vol. 3, p. 28, 1904, which appears as a supplement to ser. 2, vol. 7 of the *Archiv d. Math. u. Phys.*) how the theorem may be deduced from Weierstrass's theorem concerning finite trigonometric series (theorem VI below).

after the convergence factors have been introduced), while $F(1, \phi)$ shall by definition be $f(\phi)$. By theorems I, §2 and V, §3 this function is continuous within and on the circumference of the unit circle. Making use then of the fundamental theorem which tells us that a function continuous within and on the boundary of a region is uniformly continuous there,* we see that however small the positive quantity ϵ may be, a positive constant $r_1 < 1$ can be found such that

$$|f(\phi) - F(r, \phi)| < \epsilon \qquad \text{when} \qquad r_1 \leqq r < 1.$$

This, however, is what we mean by saying that $F(r, \phi)$ approaches $f(\phi)$ uniformly as r approaches 1.

The theorem just established enables us to prove with Picard † the following fundamental theorem due to Weierstrass:

VI. *If $f(\phi)$ is any function real and continuous for all real values of ϕ and having the period 2π, it is possible to represent it to any desired degree of approximation by means of a finite trigonometric series*

$$S_k(\phi) = \tfrac{1}{2}A_0 + \sum_1^k (A_n \cos n\phi + B_n \sin n\phi) ;$$

that is, ϵ being an arbitrarily small positive constant, it is possible to choose the integer k and the constants $A_0, \cdots A_k$ and $B_1, \cdots B_k$ in such a way that for all values of ϕ,

$$|f(\phi) - S_k(\phi)| < \epsilon.$$

* A function is said to be continuous at a point P if, no matter how small the positive quantity ϵ may be, a positive δ can be found such that if Q is any point at a distance from P less than δ (or, when P is on the boundary of the region in which we are considering our function, if Q is any point belonging to the region whose distance from P is less than δ) the values of the function at P and Q differ by less than ϵ. It is said to be continuous throughout a region if it is continuous in this sense at every point of the region. It is said to be *uniformly continuous* throughout the region if, however small the positive ϵ, a positive δ can be found such that if P and Q are any two points of the region whose distance apart is less than δ, the values of the function at these two points differ by less than ϵ. — These definitions apply to space of any number of dimensions; i. e. to functions of any number of variables. The theorem of the text is proved for the case of functions of one variable by Picard on p. 3 of the first volume of his treatise. The reader who is not familiar with this important idea is advised to study that proof and then to construct for himself along similar lines the proof for the case we need here of functions of two variables.

† See *Traité d'analyse*, 2d ed., vol. 1, p. 275, where an account of Weierstrass's original method will also be found.

To prove this we first choose a positive constant $\rho < 1$ such that for all values of ϕ

$$(22) \qquad |f(\phi) - F(\rho, \phi)| < \frac{\epsilon}{2} \cdot$$

That this is possible is seen from V. By theorem I, §2 we see that the series

$$F(\rho, \phi) = \frac{a_0}{2} + \sum_{1}^{\infty} \rho^n (a_n \cos n\phi + b_n \sin n\phi)$$

is uniformly convergent for all values of ϕ; so that, if we denote by $S_k(\phi)$ the sum of its first $k + 1$ terms:

$$(23) \qquad S_k(\phi) = \frac{a_0}{2} + \sum_{1}^{k} \rho^n (a_n \cos n\phi + b_n \sin n\phi),$$

we can then choose k so large that

$$(24) \qquad |F(\rho, \phi) - S_k(\phi)| < \frac{\epsilon}{2} \cdot$$

From the inequalities (22) and (24) follows

$$|f(\phi) - S_k(\phi)| < \epsilon,$$

and we have in the function $S_k(\phi)$, given by (23), the finite trigonometric series whose existence was affirmed in VI.

It should be carefully noticed that this approximate representation of the function $f(\phi)$ is not by any means what we should get by taking a number of terms at the beginning of the Fourier's development of $f(\phi)$, since the coefficients of $S_k(\phi)$, with the exception of the first, are not the Fourier's constants of $f(\phi)$, but are obtained by multiplying these constants by positive quantities less than 1.* We may therefore say, referring to theorem I, §1,

VII. *Weierstrass's † approximate representation of a continuous periodic function $f(\phi)$ by means of a finite trigonometric series of $k + 1$ terms is not so good an approximation, if we take the standpoint of the method of least squares, as is the sum of the first $k + 1$ terms of the Fourier's development of $f(\phi)$. It*

* It follows that if in the Fourier's expansion of $f(\phi)$ certain terms are missing, the same terms will be missing in the approximate representation of the function which we have given here.

† Formula (23) is really not Weierstrass's representation of $f(\phi)$ but Picard's. Weierstrass's differs from it in that the factors ρ^n are replaced by ρ^{n^2}. The remark here made applies, however, to both cases.

may, however, and, in some cases at least, it will be a better approximation than the sum of the first $k+1$ terms of the Fourier's development if we measure the closeness of the approximation by the size of the largest error.*

It would be of interest to determine for a given integer k the coefficients of a trigonometric series of $k+1$ terms so that it may give the best approximation in the sense just described to a given function.† So far as the writer knows, this problem has never been solved.

Without stopping to give any of the important and interesting applications of Weierstrass's theorem,‡ we turn now briefly to the case of discontinuous functions. We will consider only finite discontinuities, of which, in any finite interval, there are to be only a finite number. Let us denote the points of discontinuity in the interval $-\pi < \phi \leqq \pi$ arranged in order of magnitude by $\phi_1, \phi_2, \cdots \phi_m$ and let $\phi_1 + 2\pi = \phi_{m+1}$. Let ϵ be a positive constant.

By one of the foot-notes on p. 98 we see that it is possible to find a positive constant $\delta < 1$ such that

$$m_i - \epsilon < F(r, \phi) < M_i + \epsilon \quad \text{when} \quad \left\{ \begin{matrix} 1 - \delta < r < 1 \\ \phi_i - \delta < \phi < \phi_i + \delta \end{matrix} \right\} \; (i = 1, 2, \cdots m),$$

where m_i and M_i stand respectively for the lower and upper limits of indetermination at ϕ_i.

On the other hand the reasoning by which we established theorem V shows that for any range of values of ϕ which does not include or reach up to a point of discontinuity of $f(\phi)$ the function $F(r, \phi)$ approaches $f(\phi)$ *uniformly* as r approaches 1. Combining these facts we get at once the following theorem:

VIII. *If $f(\phi)$ has the period 2π and in any finite interval has no discontinuities other than a finite number of finite discontinuities, and if ϵ is an arbitrarily given positive constant, two positive constants $R < 1$ and δ (which last may be taken as small as we please) can be found such that when $R < r < 1$ then*

(*a*) *when the distance from ϕ to the nearest point of discontinuity is greater than or equal to δ,*

$$|f(\phi) - F(r, \phi)| < \epsilon;$$

* Certainly for all functions $f(\phi)$ whose Fourier's development does not converge at all points, or does not converge uniformly, this will occur.

† An interesting discussion of questions of this sort will be found in Klein's *Anwendung d. Diff.- u. Int.-Rechnung auf Geometrie*, Leipzig, Teubner, 1902, p. 139–171. We shall have occasion presently to consider other questions which are there treated.

‡ See Picard's treatise, 2d ed., vol. 1, p. 277–279.

and (b) when the distance from ϕ to the nearest point of discontinuity is less than or equal to δ,

$$m - \epsilon < F(r, \phi) < M + \epsilon,$$

*where m and M denote respectively the lower and upper limits of indetermination of $f(\phi)$ at this nearest point of discontinuity.**

It will be seen from this that, as r approaches 1, $F(r, \phi)$ comes as near to approaching $f(\phi)$ uniformly as it is possible for a continuous function to do which is approaching a discontinuous limit.

An immediate application of VIII is the following:

IX. *If $f(\phi)$ has the period 2π and in any finite interval has no discontinuities other than a finite number of finite discontinuities, then*

$$\lim_{r=1} \int_{c_1}^{c_2} |f(\phi) - F(r, \phi)| \, d\phi = 0.$$

For denoting by k the number of discontinuities in the interval $c_1 \leqq \phi \leqq c_2$, we can cut out from this interval k subintervals consisting of all points whose distance from one of these points of discontinuity does not exceed δ, to use the notation of VIII. In all the remaining intervals

$$|f(\phi) - F(r, \phi)| < \epsilon \qquad \text{when } R < r < 1,$$

while in these k intervals each of which is of length not exceeding 2δ,

$$|f(\phi) - F(r, \phi)| < 2K + \epsilon \qquad \text{when } R < r < 1,$$

where K is a positive constant such that for all values of ϕ

$$|f(\phi)| \leqq K.$$

Accordingly we have

$$\int_{c_1}^{c_2} |f(\phi) - F(r, \phi)| \, d\phi \leqq (c_2 - c_1)\epsilon + 2k\delta(2K + \epsilon) \qquad \text{when } R < r < 1.$$

* If we take ϕ as abscissa and y as ordinate, the curve $y = f(\phi)$ is discontinuous at all the points ϕ_i where the function $f(\phi)$ is discontinuous. Suppose now we confine our attention to the case in which all the discontinuities are finite jumps, and construct a continuous curve C by connecting by straight lines the two points whose abscissas are ϕ_i and whose ordinates are $f(\phi_i - 0)$ and $f(\phi_i + 0)$ respectively. Then it is easy to see, when we remember that by I, §2 the function $F(r, \phi)$ is continuous when $r < 1$, that VIII is equivalent to the statement that the continuous curve $y = F(r, \phi)$ approaches the continuous curve C *uniformly* as r approaches 1. That is, no matter how small the positive constant ϵ, a positive constant $R < 1$ can be found such that when $R < r < 1$, any point P being given on either one of these curves, a point Q can be found on the other such that the distance PQ is less than ϵ.

Since the right hand side of this inequality can be made as small as we please by taking ϵ and δ sufficiently small, our theorem is proved.

As an application of IX we will determine the value of the series

$$\frac{a_0^2}{2} + \sum_1^\infty (a_n^2 + b_n^2),$$

to which we were led in §1. We start from the formula

$$F(r, \phi) = \frac{a_0}{2} + \sum_1^\infty r^n (a_n \cos n\phi + b_n \sin n\phi).$$

Since this is a uniformly convergent series when r is constant and less than 1 (§2, I), it will remain uniformly convergent when multiplied by $f(\phi)$. Accordingly we have:

$$\begin{aligned}\int_{-\pi}^{\pi} & f(\phi) F(r, \phi)\, d\phi \\ &= \frac{a_0}{2}\int_{-\pi}^{\pi} f(\phi)\, d\phi + \sum_1^\infty r^n \left(a_n \int_{-\pi}^{\pi} f(\phi) \cos n\phi\, d\phi + b_n \int_{-\pi}^{\pi} f(\phi) \sin n\phi\, d\phi\right) \\ &= \pi\left\{\frac{a_0^2}{2} + \sum_1^\infty r^n (a_n^2 + b_n^2)\right\} \qquad (r < 1).\end{aligned}$$

On the right we have a power series in r which we proved in §1 to be convergent when $r = 1$. Accordingly by the theorem of Abel concerning power series which we used near the beginning of this section,

$$\frac{a_0^2}{2} + \sum_1^\infty (a_n^2 + b_n^2) = \frac{1}{\pi} \lim_{r=1} \int_{-\pi}^{\pi} f(\phi) F(r, \phi)\, d\phi \qquad (r < 1.)$$

Now we have

$$\int_{-\pi}^{\pi} f(\phi) F(r, \phi)\, d\phi = \int_{-\pi}^{\pi} \left\{f(\phi)\right\}^2 d\phi - \int_{-\pi}^{\pi} f(\phi) \left\{f(\phi) - F(r, \phi)\right\} d\phi,$$

and this last integral is easily seen to approach zero as r approaches 1. For if we denote by K a positive constant such that $|f(\phi)| \leqq K$, then

$$\left| \int_{-\pi}^{\pi} f(\phi) \left\{f(\phi) - F(r, \phi)\right\} d\phi \right| \leqq K \int_{-\pi}^{\pi} |f(\phi) - F(r, \phi)|\, d\phi.$$

Thus we have established the theorem*

X. *If a_n and b_n are the Fourier's constants of a function $f(x)$ which has the period 2π and in any finite interval has no other discontinuities than a finite number of finite discontinuities, then:*

$$(25) \qquad \frac{1}{\pi}\int_{-\pi}^{\pi} \{f(\phi)\}^2 d\phi = \frac{a_0^2}{2} + \sum_1^{\infty} (a_n^2 + b_n^2).$$

Finally by means of the same reasoning which led us from V to VI we deduce at once from VIII the theorem:

XI. *If $f(\phi)$ has the period 2π and in any finite interval has no discontinuities other than a finite number of finite discontinuities, and if ϵ is an arbitrarily given positive constant, a positive constant δ can be determined, which we are free to take as small as we please, and then we can find a positive constant $\rho < 1$ and a positive integer k such that the finite trigonometric series*

$$S_k(\phi) = \frac{a_0}{2} + \sum_1^{k} \rho^n (a_n \cos n\phi + b_n \sin n\phi)$$

(where the a's and b's are the Fourier's constants of f) has the following properties:

(*a*) *when the distance from ϕ to the nearest point of discontinuity is greater than or equal to δ,*

$$|f(\phi) - S_k(\phi)| < \epsilon;$$

(*b*) *when this distance is less than or equal to δ,*

$$m - \epsilon < S_k(\phi) < M + \epsilon,$$

where m and M denote respectively the lower and upper limits of indetermination of $f(\phi)$ at the nearest point of discontinuity.

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* Due to de la Vallée Poussin, *Annales de la Soc. scientifique de Bruxelles*, vol. 17 (1892–3), p. 18, who proves it by a method similar to that here used under the very broad restriction that $\{(fx)\}^2$ is integrable. It was rediscovered and proved by a different method by Hurwitz, *Annales de l'école normale supérieure*, vol. 19 (1902), p. 357. Both authors give also the more general form

$$(25') \qquad \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)F(\phi)d\phi = \frac{a_0A_0}{2} + \sum_1^{\infty} (a_nA_n + b_nB_n),$$

where F is a second function satisfying the same conditions as f, and where A_n and B_n are the Fourier's constants of F. This formula may, as Hurwitz remarks, be deduced immediately from (25) by applying that formula to the function $f(\phi) + F(\phi)$ instead of to $f(\phi)$.

5. La Vallee Poussin's Proof of Convergence of Fourier's Development of Continuous Functions. We will begin by establishing the following simple lemma concerning series with positive constant terms:

LEMMA. *If the series* $\sum_1^\infty u_n^2$ *converges, the series* $\sum_1^\infty \frac{|u_n|}{n}$ *will converge.*

To prove this we start from the inequality

$$\left(|u_n| - \frac{1}{n}\right)^2 \geqq 0$$

from which we infer that

$$u_n^2 + \frac{1}{n^2} \geqq \frac{2\,|u_n|}{n}.$$

Now since the series whose general term is $1/n^2$ converges, and by hypothesis the same is true of the series whose general term is u_n^2, it follows that the series whose general term is $|u_n|/n$ will also converge.

Let us now consider a continuous function $f(x)$ with period 2π. We will assume that this function has a derivative $f'(x)$ which is continuous throughout any finite interval except for a finite number of finite discontinuities. Let us denote by a_n, b_n the Fourier's constants of $f(x)$, by a_n', b_n' those of $f'(x)$. We have, on integrating by parts:

$$(26) \qquad \begin{cases} a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \dfrac{-1}{\pi n}\displaystyle\int_{-\pi}^{\pi} f'(x)\sin nx\,dx = \dfrac{-b_n'}{n}, \\ b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \dfrac{1}{\pi n}\displaystyle\int_{-\pi}^{\pi} f'(x)\cos nx\,dx = \dfrac{a_n'}{n}. \end{cases}$$

Now we know by theorems III and IV of §1 that the series

$$\sum_1^\infty a_n'^2, \quad \sum_1^\infty b_n'^2$$

converge, and from this it follows by the lemma just proved that the series

$$\sum_1^\infty \frac{|a_n'|}{n}, \quad \sum_1^\infty \frac{|b_n'|}{n}$$

converge, but these are by (26) simply the series whose general terms are $|b_n|$ and $|a_n|$, and we have the result:

I. *If $f(x)$ is continuous for all real values of x and has the period 2π, and if it has a derivative $f'(x)$ which is continuous throughout any finite interval except for a finite number of finite discontinuities, then the two series*

$$\sum_1^\infty a_n, \quad \sum_1^\infty b_n,$$

*whose terms are the Fourier's constants of $f(x)$, are absolutely convergent.**

If now we notice that the general term in the Fourier's expansion of $f(x)$ is numerically less than $|a_n| + |b_n|$, we see at once on reference to §4, theorem I, the truth of the following theorem:

II. *If $f(x)$ and $f'(x)$ satisfy the conditions imposed in* I, *then the Fourier's development of $f(x)$ converges absolutely and uniformly for all values of x to the value $f(x)$.*†

It will be seen that the only cases of continuous functions which are not covered by this theorem and which are at all likely to occur in practise are those in which $f(x)$ has an infinite derivative at some points, or at least an infinite forward or backward derivative. Such cases do occasionally occur ‡ and will be covered by a method we shall give later (§§11–13).

6. Convergence of a Special Type of Trigonometric Series. We begin by establishing the following theorem due to Schlömilch: §

* The proof we have just given shows that in the case we are considering,

$$|a_n| < \frac{C}{n}, \qquad |b_n| < \frac{C}{n},$$

where C is a positive constant. These inequalities are established by Picard (*Traité d'analyse*, 2d ed., vol. 1, p. 253) by an application of the second law of the mean *without* the assumption of the existence of a derivative $f'(x)$, but *with* the additional assumption (which we do not need) that $f(x)$ has not an infinite number of maxima and minima in any finite interval. Other interesting inequalities will also be found there; for instance, assuming $f(x)$ to be continuous, and $f'(x)$ to be continuous in any finite interval except for a finite number of finite jumps and to have at most a finite number of maxima and minima there, Picard shows that

$$|a_n| < \frac{C}{n^2}, \qquad |b_n| < \frac{C}{n^2}.$$

Cf. on this subject also Whittaker's *Modern Analysis*, p. 149–151.

† The method of proof of this theorem applies without change if the derivative $f'(x)$ is restricted merely by the requirement that $[f'(x)]^2$ be integrable. The points of discontinuity of $f'(x)$ may be everywhere dense, and $f'(x)$ need not remain finite; *cf.* la Vallée Poussin, *Annales de la Société scientifique de Bruxelles*, vol. 17 (1892–3), second part, p. 32.

‡ Cf. a paper by Kennelly on page 49 of the current volume of the ANNALS.

§ *Algebraische Analysis*, 3d ed., 1862, §31. See also *Compendium d. höheren Analysis*, vol. 1, §40, and Picard's *Traité d'analyse*, 2d ed., vol. 1, p. 251.

I. *The series*

(27) $$b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots$$

converges for all values of x provided that

1) $$\lim_{n=\infty} b_n = 0, \quad \textit{and}$$

2) *a positive integer N exists such that*

$$b_n \geqq b_{n+1} \qquad \textit{when } n \geqq N.$$

It will be noticed that according to conditions 1) and 2) the coefficients in the series beginning with b_N will be all positive, unless indeed they all vanish after a certain point.

To prove this theorem we will denote the sum of the first n terms of the series by $S_n(x)$, and write:

$$2\sin\frac{x}{2}\cdot S_n(x) = \sum_{\nu=1}^{\nu=n} 2b_\nu \sin\frac{x}{2}\sin\nu x = \sum_{\nu=1}^{\nu=n} b_\nu\left(\cos\frac{2\nu-1}{2}x - \cos\frac{2\nu+1}{2}x\right)$$

$$= b_1\cos\frac{x}{2} - (b_1 - b_2)\cos\frac{3x}{2} - (b_2 - b_3)\cos\frac{5x}{2} - \cdots$$

$$- (b_{n-1} - b_n)\cos\frac{(2n-1)x}{2} - b_n\cos\frac{(2n+1)x}{2}.$$

If x has a value for which $\sin(x/2) \neq 0$, we may write:

(28) $$S_n(x) = -b_n\frac{\cos\frac{(2n+1)x}{2}}{2\sin\frac{x}{2}} + \frac{b_1\cos\frac{x}{2} - \sum_{2}^{n}(b_{\nu-1} - b_\nu)\cos\frac{(2\nu-1)x}{2}}{2\sin\frac{x}{2}}.$$

The first fraction which stands here approaches zero as n becomes infinite, since, by 1), b_n approaches zero. It is therefore merely necessary, in order to prove (27) convergent, to show that the second fraction approaches a finite limit, i. e. that the series

(29) $$\sum_{2}^{\infty}(b_{\nu-1} - b_\nu)\cos\frac{(2\nu-1)x}{2}$$

converges. The absolute values of the terms of this series are respectively less than the absolute values of the corresponding terms of the series

$$\sum_{2}^{\infty}(b_{\nu-1}-b_\nu),$$

and since, by 1), this series is convergent, and, by 2), contains after a certain point no negative terms, it follows that (29) is absolutely convergent.

Thus the convergence of (27) is established except when $\sin(x/2)=0$, i. e. except at the points $x=\pm 2k\pi$, but at these points the series obviously converges since all its terms vanish.*

This theorem of Schlömilch may be extended as follows:†

II. *The series* (27) *whose coefficients satisfy conditions* 1) *and* 2) *converges uniformly in any interval which does not include or reach up to any of the points* $x=\pm 2k\pi$.

In order to prove this let us denote by $R_n(x)$ the remainder of the series (27) after the first n terms, so that we have by (28):

$$(30)\quad R_n(x)=\frac{b_n\cos\dfrac{(2n+1)x}{2}-\displaystyle\sum_{n+1}^{\infty}(b_{\nu-1}-b_\nu)\cos\dfrac{(2\nu-1)x}{2}}{2\sin\dfrac{x}{2}}.$$

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* By replacing x by $x+\pi$ in (27), we see that if conditions 1) and 2) of I are fulfilled, the series

$$b_1\sin x-b_2\sin 2x+b_3\sin 3x-\cdots$$

converges for all values of x. Besides these two theorems Schlömilch establishes by entirely analogous reasoning the following two:

If $\lim_{n=\infty} a_n=0$, *and a positive integer* N *exists such that*

$$a_n\geqq a_{n+1}\qquad \textit{when } n\geqq N$$

then the series:

$$\tfrac{1}{2}a_0+a_1\cos x+a_2\cos 2x+\cdots$$

converges except when $x=\pm 2k\pi$, *and the series*

$$\tfrac{1}{2}a_0-a_1\cos x+a_2\cos 2x-\cdots$$

converges except when $x=\pm(2k-1)\pi$.

† The other theorems of Schlömilch, referred to in the preceding foot-note, admit of similar extension.

Let us here take $n \geqq N$ so that b_n and all the differences $b_{\nu-1} - b_\nu$ which occur here are positive or zero, and let us denote by c a constant such that throughout the interval we are considering

$$\left| \sin \frac{x}{2} \right| \geqq c > 0.$$

Then we have throughout our interval

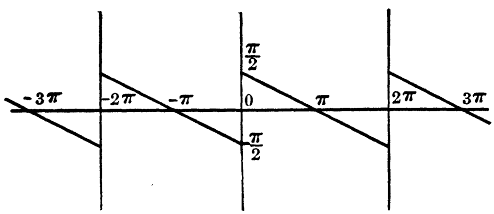
$$|R_n(x)| \leqq \frac{b_n + \sum_{n+1}^{\infty} (b_{\nu-1} - b_\nu)}{2c} = \frac{b_n}{c},$$

from which inequality, together with condition 1), the uniform convergence of (27) follows at once.

Let us now consider a special function $\Psi(x)$ defined as follows:

$$(31) \qquad \begin{cases} \Psi(x) = \dfrac{\pi - x}{2} & 0 < x < 2\pi, \\ \Psi(x + 2\pi) = \Psi(x) & x \neq 0, \pm 2\pi, \pm 4\pi, \cdots. \end{cases}$$

This function is not defined at all at the points $\pm 2k\pi$, and obviously has finite jumps of amounts π at each of these points. The curve $y = \Psi(x)$ consists of an infinite number of pieces of parallel straight lines as is indicated in the diagram. The function $\Psi(x)$ being an odd function, the same will be true of the function $\Psi(x) \cos nx$, so that all the Fourier's constants a_n are zero. On the other hand $\Psi(x) \sin nx$ being even, we have:



$$b_n = \frac{2}{\pi} \int_0^\pi \frac{\pi - x}{2} \sin nx \, dx = \frac{1}{n},$$

and the Fourier's development for $\Psi(x)$ is

$$(32) \qquad \sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots.$$

This series is of the type considered in theorems I and II.* It must therefore, by theorem I of §4, converge to the value $\Psi(x)$ at all points where this function is continuous, while at the points $x = \pm 2k\pi$ it converges (by II §4) to the value

$$\tfrac{1}{2}\Big(\Psi(x+0) + \Psi(x-0)\Big) = \tfrac{1}{2}\Big(\frac{\pi}{2} - \frac{\pi}{2}\Big) = 0,$$

as is also seen directly from the form of the series (32). Thus we have established the theorem:

III. *The Fourier's development of the function $\Psi(x)$ converges to the value $\Psi(x)$ at all points where this function is continuous, and to the value $\tfrac{1}{2}[\Psi(x+0) + \Psi(x-0)]$ at all points where it is discontinuous. It converges uniformly throughout any interval which does not include or reach up to a point of discontinuity.*†

7. Convergence of the Fourier's Expansion of Functions with Finite Jumps. We can generalize from the special function of the last section by successive steps.‡

1) We consider the function

$$\Psi_a(x) = \Psi(x-a) \qquad (-\pi < a \leqq \pi),$$

which is continuous except at the points $x = a,\ a \pm 2\pi,\ a \pm 4\pi, \cdots$ at each of which it has a jump of amount π. Replacing x by $x - a$ in (32) we find:

$$\Psi_a(x) = \sum_{1}^{\infty}\Big(-\frac{\sin na}{n}\cos nx + \frac{\cos na}{n}\sin nx\Big) \tag{33}$$

except at the points of discontinuity where the series has the value $\tfrac{1}{2}[\Psi_a(x+0) + \Psi_a(x-0)]$. Moreover, as we see by theorem VII of §1,

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* We note in passing that such a series as

$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots$$

would *not* be of this type.

† This fact may easily be proved, without making use of theorems I and II, by using a formula for the sum of the first n terms of (32) which will be found below in formula (47). Cf. Kneser: *Sitzungsberichte d. Berl. math. Ges.*, Feb. 1904, p. 33. This is an Appendix to *Archiv d. Math. u. Phys.*, ser. 3, vol. 7.

‡ The fundamental idea is here analogous to that used by Kneser in the paper referred to in the last foot-note. The details, however, are wholly different and decidedly simpler.

formula (33) gives the Fourier's development of $\Psi_a(x)$. Thus we see that theorem III of §6 holds without change if $\Psi(x)$ is replaced by $\Psi_a(x)$.

2) The function

$$\frac{\lambda}{\pi}\,\Psi_a(x)$$

differs from $\Psi_a(x)$ only in having finite jumps of magnitude λ instead of jumps of magnitude π. Its Fourier's development will (by VI, §1) be obtained by multiplying (33) by the constant λ/π. Thus if this function be put in place of Ψ, theorem III of §6 will still hold.

3) Let $a_1, \cdots a_k$ be arbitrarily chosen points in the interval $-\pi < x \leqq \pi$, and $\lambda_1, \cdots \lambda_k$ arbitrarily chosen constants, and consider the function

$$F(x)=\frac{\lambda_1}{\pi}\,\Psi_{a_1}(x)+\cdots+\frac{\lambda_k}{\pi}\,\Psi_{a_k}(x). \tag{34}$$

This function is continuous everywhere except at the points $a_1, \cdots a_k$ and points differing from them by multiples of 2π, at which points it has finite jumps of magnitudes $\lambda_1, \cdots \lambda_k$ respectively. The Fourier's development of $F(x)$ will, by theorem V, §1, be obtained by adding together the Fourier's developments of the functions $\frac{\lambda_i}{\pi}\,\Psi_{a_i}(x)$. These being of the form considered under 2), we see that theorem III of §6 holds if for $\Psi(x)$ is substituted $F(x)$. Moreover we note that $F(x)$ has a derivative at every point where it is continuous, and that this derivative has the value zero,* since this is true of the functions $\Psi_a(x)$.

4) Finally let us consider *any* function $f(x)$ of period 2π which in any finite interval has no other discontinuities than a finite number of finite jumps, and which has a first derivative which has in any finite interval only a finite number of finite discontinuities. Denote by $a_1, \cdots a_k$ the points of discontinuity of $f(x)$ which lie in the interval $-\pi < x \leqq \pi$, and by $\lambda_1, \cdots \lambda_k$ the amounts of the jumps at these points. The function $F(x)$ having the same meaning as in 3), let us consider the function

$$\Phi(x)=f(x)-F(x). \tag{35}$$

This function has the period 2π and is continuous except at the points where f and F are discontinuous. Moreover, since at each of these points f and F

* Graphically this means that the curve $y=F(x)$ consists of pieces of straight lines.

have finite jumps of the same magnitude, Φ has a finite jump there of magnitude zero, i. e. the discontinuity of Φ is due merely to the fact that it is not defined at this point. Let us then *add* to the definition of Φ by defining it at each point where f is discontinuous to have the value $\Phi(x+0)=\Phi(x-0)$. The function Φ will then be everywhere continuous, and its derivative will have in any finite interval at most a finite number of finite discontinuities. It follows from theorem II of §5 that the Fourier's development of $\Phi(x)$ converges uniformly to the value $\Phi(x)$ for all values of x. Since the Fourier's development of $F(x)$ converges by 3) to the value $F(x)$ uniformly throughout any interval not including or reaching up to any of its points of discontinuity, and since we have for all points where $f(x)$ is continuous:

$$f(x) = \Phi(x) + F(x),$$

it follows from theorem V of §1 that the Fourier's development of $f(x)$ converges uniformly to the value $f(x)$ throughout any interval which does not include or reach up to any discontinuity of $f(x)$. Finally at a point of discontinuity of f the function F is also discontinuous, but its Fourier's development converges, as we saw in 3), to the value $\frac{1}{2}[F(x+0)+F(x-0)]$. At such a point the Fourier's development of $\Phi(x)$ converges to the value $\Phi(x)$, or, as we may write it if we choose, $\frac{1}{2}[\Phi(x+0)+\Phi(x-0)]$. Accordingly, using theorem V of §1 again, the Fourier's development of $f(x)$ converges here to the value $\frac{1}{2}[f(x+0)+f(x-0)]$.* We have thus established the theorem:

I. *If $f(x)$ has the period 2π and in any finite interval has no other discontinuities than a finite number of finite jumps, and if $f(x)$ has a derivative which in any finite interval has no other discontinuities than a finite number of finite discontinuities,† then the Fourier's development of $f(x)$ converges to the value $f(x)$ at all points where f is continuous, and to the value $\frac{1}{2}[f(x+0)+f(x-0)]$ when f is discontinuous. Moreover the convergence is uniform throughout any interval which does not include or reach up to a point of discontinuity of f.*

* It should be noted that we have avoided making use of theorem II of §4 since the simplification to be gained by such use would be insignificant.

† The proof applies without change to the more general case in which we merely require that $[f'(x)]^2$ be integrable throughout any finite interval. Cf. the foot-note to theorem II of §5.

8. The Differentiation and Integration of Fourier's Series. We will take as our starting point the following purely formal theorem which makes no reference to the convergence of the series.

I. *If $f(x)$ is a continuous function with period 2π and has a derivative $f'(x)$ which in any finite interval is continuous except for a finite number of finite discontinuities,* then the Fourier's development of $f'(x)$ may be obtained by differentiating term by term the Fourier's development of $f(x)$.*

This theorem has practically been already proved in §5 (see formula (26)), for we saw there that if a_n, b_n are the Fourier's constants of $f(x)$ and a'_n, b'_n those of $f'(x)$ then :

$$a'_n = n\,b_n, \qquad b'_n = -\,n\,a_n,$$

and these are precisely the coefficients in the series we obtain by differentiating term by term the Fourier's development of $f(x)$.

This theorem may be made more than merely a formal one by imposing on the function $f(x)$ further conditions which will secure the convergence of the Fourier's development of $f'(x)$.† We thus obtain, by applying the theorem a number of times, the following result:

II. *If the continuous function $f(x)$ with period 2π has continuous derivatives of the first $n-1$ orders $f'(x), \cdots f^{[n-1]}(x)$, and a derivative of the n^{th} order $f^{[n]}(x)$ which is continuous throughout any finite interval except for a finite number of finite discontinuities, then the Fourier's development of $f(x)$ may be differentiated term by term $n-1$ times, and, provided $f^{[n]}(x)$ admits a convergent Fourier's development, it may be differentiated term by term n times.*

Before going further with the subject of differentiation let us turn to the subject of the integration. The simplest case would be that in which $f(x)$ is continuous and has a derivative which in any finite interval has only a finite number of finite discontinuities. The Fourier's development of such a function (being uniformly convergent by theorem II §5) may be integrated term by term between any two finite limits. When we have to deal with the more general case referred to in theorem I of §7 this is no longer so obvious if we

* This theorem remains true if f' is finite and integrable in Riemann's sense, or even if f' is not finite, but $|\,f'(x)\,|$ is integrable. A corresponding generalization may be made in the subsequent theorems.

† The conditions already imposed are, by §5 theorem II, enough to secure the convergence of the Fourier's development for $f(x)$.

wish to integrate up to or over a point of discontinuity of $f(x)$. A closer examination, which we shall make in the next section, of the nature of the convergence near such a point of discontinuity would enable us to infer here also that we have a right to integrate term by term. These are, however, merely special cases of the following far more general theorem due to la Vallée Poussin (l. c., p. 32);

III. *If $f(x)$ has the period 2π, and in any finite interval is continuous except for a finite number of finite discontinuities, then, although the Fourier's development of $f(x)$*

$$\tfrac{1}{2} a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx)$$

need not converge, the integral of $f(x)$ between any two finite limits may be obtained by integrating this series term by term between these limits, and, if $F(x)$ denotes the general continuous indefinite integral of $f(x)$, it will be given by the convergent series

$$(36) \qquad F(x) = C + \tfrac{1}{2} a_0 x + \sum_1^\infty \left(\frac{a_n}{n} \sin nx - \frac{b_n}{n} \cos nx\right).$$

For we may write $F(x) = \int_{-\pi}^{x} f(x)\, dx + c$, and therefore we have

$$F(x + 2\pi) = F(x) + \int_{-\pi}^{\pi} f(x) dx = F(x) + \pi a_0.$$

Accordingly $F(x) - \tfrac{1}{2} a_0 x$ has the period 2π, and since it is continuous and has as its derivative $f(x) - \tfrac{1}{2} a_0$, it can be developed in a convergent Fourier's series:

$$(37) \qquad F(x) - \tfrac{1}{2} a_0 x = C + \sum_1^\infty (A_n \cos nx + B_n \sin nx).$$

By theorem I, the result of differentiating this series term by term is to give the Fourier's development of $f(x) - \tfrac{1}{2} a_0$, that is the series

$$\sum_1^\infty (a_n \cos nx + b_n \sin nx).$$

Accordingly

$$A_n = -\frac{b_n}{n}, \quad B_n = \frac{a_n}{n}.$$

Substituting these values, formula (37) reduces to (36), which is thus established.

The remaining part of our theorem, concerning the definite integral of $f(x)$, now follows at once. For the terms of (36) are indefinite integrals of the corresponding terms of the Fourier's development of $f(x)$, so that if in the equation

$$\int_a^\beta f(x)dx = F(\beta) - F(a)$$

we replace $F(\beta)$ and $F(a)$ by their values from (36), the second member reduces to the series formed by integrating term by term from a to β the Fourier's development of $f(x)$. Thus our theorem is proved.*

---

* A generalization of this theorem which is often useful is the following:

III′ *If $f(x)$ and $\phi(x)$ are finite and integrable functions, and f has the period 2π, the integral between any two finite limits of the product of $f\phi$ may be obtained by multiplying the Fourier's development of f by ϕ, and then integrating term by term.*

In proving this theorem it will be sufficient to consider the case in which the distance between the limits of integration does not exceed 2π, since a longer interval could be cut up into pieces each of which is of a length not greater than 2π. We may even suppose the distance between the limits of integration exactly equal to 2π, for if this length were less than 2π we could regard $\phi(x)$ as having the value zero everywhere outside of the interval of integration; and we could then, without affecting the result, extend our interval of integration until it is of length 2π. Finally we may obviously suppose the integration to take place in the positive direction, the opposite case being at once reducible to this. Thus we have finally to establish the formula

$$\int_c^{c+2\pi} f(x)\phi(x)dx = \tfrac{1}{2}\, a_0 \int_c^{c+2\pi} \phi(x)dx + \sum_1^\infty \int_c^{c+2\pi} (a_n \cos nx + b_n \sin nx)\phi(x)dx,$$

where the a's and b's are the Fourier's constants of $f(x)$. If now we regard ϕ as defined outside of the interval $c \leqq x < c + 2\pi$ so that it has the period 2π, and denote its Fourier's constants by α_n and β_n, then, remembering that f and ϕ both have the period 2π, the formula we wish to establish reduces to

$$\int_{-\pi}^{\pi} f(x)\phi(x)dx = \pi[\tfrac{1}{2}\, a_0\alpha_0 + \sum_1^\infty (a_n\alpha_n + b_n\beta_n)],$$

and this is precisely the generalization of formula (25) §4 which was established in the foot note on p. 107.

This proof shows at once that the restriction that f and ϕ be finite was not necessary. It is sufficient to require that $\{f(x)\}^2$ and $\{\phi(x)\}^2$ be integrable. There are, however, other cases of some importance to which the proof here given does not apply, but in which the theorem remains true. An application of the results of §9 shows that this will be the case if $|\phi(x)|$ is integrable and $f(x)$ satisfies the conditions of theorem I §7.

We are now in a position to see clearly the necessity in theorem I for the requirement that $f(x)$ be continuous. If in that theorem $f(x)$ had finite discontinuities, we could reason as follows. If by differentiating the Fourier's development of $f(x)$ we did obtain the Fourier's development of $f'(x)$, then this development would contain no constant term, and therefore by theorem III the Fourier's development of $f(x)$ would represent a *continuous* indefinite integral of $f'(x)$; but this is possible only if $f(x)$ is either continuous or has no other discontinuities than finite jumps of magnitude zero (see the first foot-note on p. 89). We thus have the result:

IV. *If $f(x)$ has the period 2π, and in any finite interval $f(x)$ and its derivative $f'(x)$ are continuous except for a finite number of finite discontinuities, then the series obtained by differentiating the Fourier's development of $f(x)$ will be the Fourier's development of $f'(x)$ when and only when $f(x)$ has no discontinuities except removable discontinuities.*

Although this theorem shows us that by differentiating term by term the Fourier's development of a discontinuous function $f(x)$ we shall not be led to the Fourier's development of $f'(x)$ it would still be conceivable that the trigonometric series to which we are led might converge to the value $f'(x)$. It would be possible to prove that, at least in the simpler cases, this will not occur. The result being, however, of a merely negative character we will not stop to establish it here.

A wholly different problem presents itself to us if we start from a trigonometric series without knowing anything about the function which it represents, or even that the series is a Fourier's series, and wish first to ascertain whether the function represented by the series has a derivative; and second, if such a derivative exists to find an expression for it in the form of a series. The most obvious answer to the question which here presents itself is given by the following theorem, which is merely an application to this case of one of the fundamental theorems in the theory of uniform convergence.

V. *If the trigonometric series*

$$(38) \qquad \sum_{1}^{\infty}(a_n \cos nx + b_n \sin nx)^{*}$$

---

* From here on we take the series without constant term since the presence of such a term would obviously have no effect on our theorems.

converges for a particular value c of x, and if the series

$$(39)\qquad \sum_{1}^{\infty}(-n\,a_n \sin nx + n\,b_n \cos nx),$$

obtained by differentiating it term by term, converges uniformly throughout an interval $A \leqq x \leqq B$ which includes the point c, then (38) *will converge throughout this interval* and the function $f(x)$ represented by it has throughout this interval a derivative represented by* (39).

In the case of many of the simplest and most important series, however, (cf. the series in (32) above) this theorem fails to give us any information, as the series (39) diverges except at isolated points. We proceed therefore to establish a further criterion due to Lerch.†

We again assume that (38) converges when $x = c$, and we will suppose that c does not have any of the values, $0, \pm\pi, \pm 2\pi, \cdots$; and we consider an interval $A \leqq x \leqq B$ which includes the point c but does not include any of the points $\pm n\pi$. Let us denote by $S_k(x)$ the sum of the first k terms of (38), and by $S'_k(x)$ the derivative of $S_k(x)$, i. e. the sum of the first k terms of (39). Consider now the function‡

$$
\begin{aligned}
(40)\qquad 2\sin x\, S'_k(x) &= \sum_{1}^{k}\Big\{-2na_n \sin nx \sin x + 2nb_n \cos nx \sin x\Big\}\\
&= \sum_{1}^{k}\Big\{na_n[\cos(n+1)x - \cos(n-1)x]\\
&\qquad\qquad + nb_n[\sin(n+1)x - \sin(n-1)x]\Big\}\\
&= \sum_{0}^{k-1}\Big\{[(n-1)a_{n-1} - (n+1)a_{n+1}]\cos nx\\
&\qquad\qquad + [(n-1)b_{n-1} - (n+1)b_{n+1}]\sin nx\Big\} + R_k(x),
\end{aligned}
$$

where for convenience we write $a_{-1} = b_{-1} = a_0 = b_0 = 0$, and

$$
\begin{aligned}
(41)\qquad R_k(x) = {}&(k-1)a_{k-1}\cos kx + (k-1)b_{k-1}\sin kx\\
&+ ka_k \cos(k+1)x + kb_k \sin(k+1)x.
\end{aligned}
$$

* We may add, if we wish, that it will converge uniformly there.

† *Annales de l'Ecole Normale supérieure*, Ser. 3, vol. 12 (1895), p. 351.

‡ The identity of the method here used with that employed by Schlömilch (cf. §6) will be at once obvious.

If the conditions of theorem V were fulfilled, $R_k(x)$ would approach zero uniformly as k becomes infinite,* and $S'_k(x)$ would approach uniformly the derivative $f'(x)$ of the function represented by (38). Accordingly the series

$$(42)\quad \sum_0^\infty \left\{ [(n-1)a_{n-1}-(n+1)a_{n+1}]\cos nx + [(n-1)b_{n-1}-(n+1)b_{n+1}]\sin nx \right\}$$

would converge uniformly to the value $2\sin x f'(x)$. This will also be true in many cases in which the conditions of theorem V are not fulfilled, as we will now show.

Let us assume that the series (42) converges uniformly throughout the interval $A \leqq x \leqq B$. The same will therefore be true of the series

$$\sum_0^\infty \left\{ \Big[(n-1)a_{n-1}-(n+1)a_{n+1}\Big]\frac{\cos nx}{2\sin x} + \Big[(n-1)b_{n-1}-(n+1)b_{n+1}\Big]\frac{\sin nx}{2\sin x} \right\}.$$

If we denote by $g(x)$ the continuous function represented by this series we therefore have

$$(43)\quad \int_c^x g(x)\,dx = \lim_{k=\infty} \sum_0^{k-1} \int_c^x \left\{ \Big[(n-1)a_{n-1} - (n+1)a_{n+1}\Big]\frac{\cos nx}{2\sin x} + \Big[(n-1)b_{n-1} - (n+1)b_{n+1}\Big]\frac{\sin nx}{2\sin x} \right\} dx.$$

Let us now consider the integral

$$\int_c^x \frac{R_k(x)}{2\sin x}\,dx = \sum_k^{k+1} \left\{ (n-1)a_{n-1}\int_c^x \frac{\cos nx}{2\sin x}\,dx + (n-1)b_{n-1}\int_c^x \frac{\sin nx}{2\sin x}\,dx \right\}.$$

Integrating by parts we may write

$$(44)\quad \begin{cases} \displaystyle\int_c^x \frac{\cos nx}{\sin x}\,dx = \frac{\sin nx}{n\sin x} - \frac{\sin nc}{n\sin c} + \frac{1}{n}\int_c^x \frac{\cos x\sin nx}{\sin^2 x}\,dx, \\ \displaystyle\int_c^x \frac{\sin nx}{\sin x}\,dx = -\frac{\cos nx}{n\sin x} + \frac{\cos nc}{n\sin c} - \frac{1}{n}\int_c^x \frac{\cos x\cos nx}{\sin^2 x}\,dx. \end{cases}$$

---

* For (39) being uniformly convergent, its coefficients ka_k and kb_k would approach zero as k becomes infinite (cf. I§14, foot-note).

Now introduce a positive constant K such that throughout our interval $\sin^2 x > K$, and remember that the length of our interval is $< \pi < 4$. We then see that both integrals (44) are in absolute value less than $6/(nK)$. Accordingly we may write

$$\left| \int_c^x \frac{R_k(x)}{2 \sin x} dx \right| < \frac{3}{K} \left\{ |a_{k-1}| + |b_{k-1}| + |a_k| + |b_k| \right\}.$$

If, then, we add to our other assumptions the further one that a_k and b_k approach zero as k becomes infinite, we get the result:

(45) $$\lim_{k=\infty} \int_c^x \frac{R_k(x)}{2 \sin x} dx = 0.*$$

Let us now return to formula (40), and, after dividing both sides by $2 \sin x$, integrate it from c to any point x in the interval $A \leqq x \leqq B$. The expression which we thus get on the right is seen, by (43) and (45), to approach $\int_c^x g(x) dx$ as its limit as k becomes infinite. The same must therefore be true of $S_k(x) - S_k(c)$ which is what stands on the left. But, since we have assumed that (38) converges when $x = c$, or in other words that $S_k(c)$ approaches a finite limit as k becomes infinite, it follows that $S_k(x)$ must also approach a finite limit when k becomes infinite. Denoting this value, that is the value of (38), by $f(x)$ we thus have

$$f(x) - f(c) = \int_c^x g(x) dx.$$

From this equation we infer that $f(x)$ has a derivative, and that this derivative is $g(x)$. Thus we have established the following theorem:

VI. *If the trigonometric series* (38) *converges for a particular value c of x which is not an integral multiple of* π, *and if*

$$\lim_{n=\infty} a_n = 0, \qquad \lim_{n=\infty} b_n = 0,$$

* The inequality last written shows that this limit is approached uniformly for all values of x in the interval $A \leqq x \leqq B$.

then throughout an interval $A \leqq x \leqq B$ which includes c but does not include any integral multiple of π, the series (38) will converge and the function $f(x)$ represented by it will have a derivative $f'(x)$ given by the equation*

$$2 \sin x\, f'(x) = \sum_{0}^{\infty} \Big\{ [(n-1)\, a_{n-1} - (n+1)\, a_{n+1}] \cos nx + [(n-1)\, b_{n-1} - (n+1)\, b_{n+1}] \sin nx \Big\},$$

(where $a_{-1} = b_{-1} = a_0 = b_0 = 0$), provided the series just written converges uniformly throughout the interval $A \leqq x \leqq B$.

That this theorem is really more general than V will be seen by applying it to the special series (32) and to many other similar ones. For further applications the article of Lerch above cited should be consulted.

9. The Character of the Convergence near a Finite Jump. Gibbs's Phenomenon. In an interval which includes or reaches up to a point where $f(x)$ is discontinuous the Fourier's development of $f(x)$ cannot converge uniformly, since a uniformly convergent series of continuous functions necessarily represents a continuous function. Confining our attention to a point where $f(x)$ has a finite jump, we wish in this section to get as clear an idea as possible of the nature of the convergence in the neighborhood of this point.

For this purpose we begin by considering the special function $\Psi(x)$ of §6.† We will denote by $S_n(x)$ the sum of the first n terms of the Fourier's expansion (32) of $\Psi(x)$:

$$S_n(x) = \sin x + \frac{\sin 2x}{2} + \cdots + \frac{\sin nx}{n}$$
$$= \int_0^x [\cos \alpha + \cos 2\alpha + \cdots + \cos n\alpha]\, d\alpha.$$

* We may add, if we wish, that (38) converges uniformly throughout this interval. The proof just given may be readily seen to establish this fact also, since, as was remarked in the last foot-note, the limit in (45) is approached uniformly, and the same is true of the limit in (43), since the integral from c to x of a uniformly convergent series is itself uniformly convergent.

† For a similar treatment of another special function cf. Runge's book *Theorie und Praxis der Reihen*, Leipzig, 1904, pp. 170–180.

If we let $z = \cos a + i \sin a$, the sum of cosines which stands here under the integral sign is the real part of the geometric progression

$$z + z^2 + \cdots + z^n = \frac{z - z^{n+1}}{1 - z}.$$

By an easy trigonometric reduction we thus establish the formula

$$\cos a + \cos 2a + \cdots + \cos na = -\tfrac{1}{2} + \frac{\sin(n + \frac{1}{2})a}{2 \sin \frac{a}{2}}. \tag{46}$$

Substituting this above gives

$$S_n(x) = -\frac{x}{2} + \int_0^x \frac{\sin(n + \frac{1}{2})a}{2 \sin \frac{a}{2}} da. \tag{47}$$

We will denote by $R_n(x)$ the remainder of the series (32) after the n^{th} term. If we confine our attention to the interval $0 < x < 2\pi$, in which $\Psi(x) = \frac{1}{2}(\pi - x)$, we may write

$$R_n(x) = \frac{\pi}{2} - \int_0^x \frac{\sin(n + \frac{1}{2})a}{2 \sin \frac{a}{2}} da \qquad (0 < x < 2\pi). \tag{48}$$

Let us now enquire at what points the approximation to the function $\Psi(x)$ given by $S_n(x)$ is worst, i. e. for what points $R_n(x)$ has a positive maximum or a negative minimum, n being regarded as fixed. Differentiating (48) we find

$$R'_n(x) = -\frac{\sin(n + \frac{1}{2})x}{2 \sin \frac{x}{2}} \qquad (0 < x < 2\pi). \tag{49}$$

This expression vanishes when and only when x has one of the following values:

$$x_1 = \frac{2\pi}{2n+1}, \quad x_2 = \frac{4\pi}{2n+1}, \quad x_3 = \frac{6\pi}{2n+1}, \quad \cdots \quad x_{2n} = \frac{4n\pi}{2n+1}. \tag{50}$$

By considering the sign of $R'_n(x)$ for points on both sides of each of these points x_i we see that $R_n(x)$ has minima at the points x_1, x_3, x_5, $\cdots$, maxima

at the points $x_2, x_4, x_6, \cdots$. In order to show that $|R_n(x)|$ has maxima at all of these points it would be necessary to prove that $R_n(x)$ is negative at all the points $x_1, x_3, x_5, \cdots$, positive at all the points $x_2, x_4, x_6 \cdots$. This is true,* but we will not stop to prove it, since we need to know it merely for large values of n, and that we shall learn in the course of our work.

We will now throw $R_n(x)$ into a different form. For this purpose, modify the expression (48) for $R_n(x)$ by adding and subtracting the quantity

$$\int_0^x \frac{\sin(n+\frac{1}{2})a}{a}\,da = \int_0^{(n+\frac{1}{2})x} \frac{\sin x}{x}\,dx.$$

We thus get

$$(51) \qquad R_n(x) = \frac{\pi}{2} - \int_0^{(n+\frac{1}{2})x} \frac{\sin x}{x}\,dx + I_n(x),$$

where

$$(52) \qquad I_n(x) = \int_0^x \frac{2\sin\frac{1}{2}a - a}{2a\sin\frac{1}{2}a}\sin(n+\tfrac{1}{2})a\,da.$$

An integration by parts gives

$$I_n(x) = \frac{x - 2\sin\frac{1}{2}x}{2x\sin\frac{1}{2}x} \cdot \frac{\cos(n+\frac{1}{2})x}{n+\frac{1}{2}} + \frac{1}{n+\frac{1}{2}}\int_0^x \frac{a^2\cos\frac{1}{2}a - 4\sin^2\frac{1}{2}a}{4a^2\sin^2\frac{1}{2}a}\cos(n+\tfrac{1}{2})a\,da.$$

The two functions

$$\frac{x - 2\sin\frac{1}{2}x}{2x\sin\frac{1}{2}x}, \qquad \frac{x^2\cos\frac{1}{2}x - 4\sin^2\frac{1}{2}x}{4x^2\sin^2\frac{1}{2}x},$$

which occur in this expression, both become infinite when $x = 2\pi$. If we confine our attention to an interval $0 \leqq x \leqq b$ where $b < 2\pi$, we see that both these functions are continuous throughout this interval except at the point $x = 0$

* The proof follows at once from theorem IX of §10 according to which $R_n(x)$ changes sign at least $2n+2$ times in the interval $0 \leqq x \quad 2\pi$. Since $R_n(+0) = \frac{1}{2}\pi$, $R_n(-0) = -\frac{1}{2}\pi$, $R_n(x)$ must change sign at least $2n+1$ times in the interval $0 < x < 2\pi$. It cannot change sign more than once in any of the intervals $0x_1$, $x_1 x_2, \ldots x_{2n}, 2\pi$, for otherwise $R'_n(x)$ would vanish in this interval. Accordingly $R_n(x)$ changes sign just once between 0 and x_1, once between x_1 and x_2, etc. Since $R_n(+0)$ is positive it follows that $R_n(x)$ is alternately negative and positive at the points $x_1, x_2, \ldots x_{2n}$.

where they are not defined, and that they both approach finite limits as x approaches zero. It is therefore possible to find a positive constant M such that

$$\left|\frac{x - 2\sin\frac{1}{2}x}{2x\sin\frac{1}{2}x}\right| < M, \quad \int_0^x \left|\frac{a^2\cos\frac{1}{2}a - 4\sin^2\frac{1}{2}a}{4a^2\sin^2\frac{1}{2}a}\right| da < M, \qquad (0 < x < b,\ b < 2\pi).$$

We thus obtain the inequality

$$\left|I_n(x)\right| < \frac{2M}{n+\frac{1}{2}} \qquad (0 < x < b,\ b < 2\pi).$$

This inequality shows that $I_n(x)$ approaches zero as n becomes infinite not merely if x remains fixed but also if x is allowed to vary in any manner whatever with n, subject merely to the inequality $0 < x < b$.

Let us now go back to formula (51) and use it to express the value of R_n at the points (50):

$$(53) \qquad R_n(x_k) = \frac{\pi}{2} - \int_0^{k\pi} \frac{\sin x}{x}\,dx + I_n\left(\frac{2k\pi}{2n+1}\right).$$

Here we regard k as a given positive integer. If now we allow n to become infinite, the first two terms remain constant while the last term, as we have just seen, approaches zero. Accordingly we may write

$$(54) \qquad P_k = \lim_{n=\infty} R_n(x_k) = \frac{\pi}{2} - \int_0^{k\pi} \frac{\sin x}{x}\,dx.$$

We now recall the well known formula *

$$(55) \qquad \frac{\pi}{2} = \int_0^{+\infty} \frac{\sin x}{x}\,dx = u_0 + u_1 + u_2 + \cdots,$$

where

$$(56) \qquad u_i = \int_{i\pi}^{(i+1)\pi} \frac{\sin x}{x}\,dx \qquad (i = 0, 1, 2, \cdots);$$

and we notice that the quantity P_k defined in (54) is merely the remainder of the series (55) after the first k terms:

$$(57) \qquad P_k = u_k + u_{k+1} + u_{k+2} + \cdots.$$

---

* See for instance Jordan, *Cours d'analyse*, vol. 2, p. 104, or Picard, *Traité d'analyse*, vol. 2, p. 168 (1st edition, p. 156).

From their definition (56), we see that the constants u_i are positive when i is even, negative when i is odd and that they satisfy the inequality

$$(58)\quad \frac{2}{(i+1)\pi} = \left| \int_{i\pi}^{(i+1)\pi} \frac{\sin x}{(i+1)\pi}\, dx \right| < |u_i| < \left| \int_{i\pi}^{(i+1)\pi} \frac{\sin x}{i\pi}\, dx \right| = \frac{2}{i\pi},$$

so that $$|u_i| < |u_{i-1}| \quad \text{and} \quad \lim_{i=\infty} u_i = 0.$$

The series (55) therefore belongs to the simple class of series whose terms

1) are alternately positive and negative,
2) continually decrease in numerical value,
3) approach the limit zero.

In such a series it is well known, and at once evident, that the remainder P_k has the same sign as u_k, so that P_k is positive when k is even, negative when k is odd. The same will therefore be true of $R_n(x_k)$, at least for large values of n, as was stated above.

Moreover since

$$P_k = u_k + u_{k+1} + P_{k+2},$$

and since $u_k + u_{k+1}$ has the same sign as P_{k+2}, we have

$$|P_k| = |u_k + u_{k+1}| + |P_{k+2}|,$$

and accordingly

$$(59)\qquad |P_k| > |P_{k+2}|^* \qquad (k = 1, 2, \cdots).$$

---

* It is in fact true that

$$|P_k| > |P_{k+1}|,$$

as is fairly obvious from the numerical values given below. To prove this, however, it is necessary to get for the u's somewhat closer inequalities than those contained in formula (58). Such inequalities, which, as we shall see in a moment, are also useful for purposes of numerical computation, may be obtained as follows.

Throughout the interval $k\pi < x < (k+1)\pi$ the ordinates of the equilateral hyperbola $y = 1/x$ are smaller than the corresponding ordinates of the chord which joins the two ends of this arc, and greater, except at the point of contact, than the ordinates of the tangent drawn at the point of the curve whose abscissa is $(k+\frac{1}{2})\pi$. Hence:

$$\left.\begin{aligned} \frac{1}{x} &< \frac{-x}{k(k+1)\pi^2} + \frac{2k+1}{k(k+1)\pi} \\ \frac{1}{x} &\geqq \frac{-4x}{(2k+1)^2\pi^2} + \frac{4}{(2k+1)\pi} \end{aligned}\right\} \quad k\pi < x < (k+1)\pi.$$

The numerical values of the constants $u_0, u_1, \cdots$ may be computed to any desired degree of accuracy by methods of mechanical quadrature or other methods of approximation.* We thus find:

$$\begin{array}{llll} u_0 = & 1.8519, & u_4 = & 0.142, \\ u_1 = & -0.434, & u_5 = & -0.116, \\ u_2 = & 0.257, & u_6 = & 0.098, \\ u_3 = & -0.183, & & \end{array}$$

while from here on the values are given to four places of decimals by the following formula (cf. the two preceeding foot-notes):

$$u_k = \frac{4}{(2k+1)\pi} + \epsilon_k \qquad (0 < \epsilon_k < 0.0001 \text{ when } k \geqq 7).$$

Accordingly:

$$(\mathrm{A})\begin{cases} |u_k| < \left|\int_{k\pi}^{(k+1)\pi}\left(\frac{-x}{k(k+1)\pi^2} + \frac{2k+1}{k(k+1)\pi}\right)\sin x\,dx\right| = \frac{2k+1}{k(k+1)\pi} & (k=1,2,\ldots), \\ |u_k| > \left|\int_{k\pi}^{(k+1)\pi}\left(\frac{-4x}{(2k+1)^2\pi^2} + \frac{4}{(2k+1)\pi}\right)\sin x\,dx\right| = \frac{4}{(2k+1)\pi} & (k=0,1,2,\ldots). \end{cases}$$

From these inequalities we readily deduce the following ones $(i = 0, 1, 2, \ldots)$:

$$(|u_i| - |u_{i+1}|) - (|u_{i+1}| - |u_{i+2}|) > \frac{12i + 18}{(i+1)(i+2)(2i+1)(2i+5)\pi} \quad 0.$$

Now:

$$|P_k| = \sum_{\nu=0}^{\nu=\infty}(|u_{k+2\nu}| - |u_{k+2\nu+1}|), \quad |P_{k+1}| = \sum_{\nu=0}^{\nu=\infty}(|u_{k+2\nu+1}| - |u_{k+2\nu+2}|),$$

and, since by the last written inequality every term in the first series is greater than the corresponding term in the second, it follows that $|P_k| > |P_{k+1}|$.

* The computation of u_0 may be readily effected by using the Maclaurin's development of $\sin x$. A fair approximation for the subsequent u's is given by the inequalities (A) of the preceeding foot-note. A decidedly better result is obtained by replacing the first of these inequalities by:

$$|u_k| < \frac{2k+1}{k(k+1)\pi} - \frac{2}{k(k+1)(2k+1)\pi^2} \qquad (k = 1, 2, \cdots),$$

a formula obtained by replacing, in the interval $k\pi < x < (k+1)\pi$, the arc of the equilateral hyperbola by the broken line formed by joining the extremities of this arc with the point on it whose abscissa is $(k+\frac{1}{2})\pi$. [*Note added Feb. 3, 1906:* In the January number of the *American Mathematical Monthly* (p. 12), which has just been issued, I find all but the last of the numerical values which I have here given worked out by a wholly different method by S. A. Corey. The values are there given to six decimal places.]

Starting from the value $P_0 = \frac{1}{2}\pi$, and using the relation $P_k = u_k + P_{k+1}$, we readily get the values:

$$\begin{aligned} P_0 &= 1.5708, & P_4 &= 0.079, \\ P_1 &= -0.2811, & P_5 &= -0.063, \\ P_2 &= 0.153, & P_6 &= 0.053, \\ P_3 &= -0.104, & P_7 &= -0.045. \end{aligned}$$

A little reflection will show the bearing of these figures on the question as to how the function $\Psi(x)$ is approximated to near the origin by the first n terms of its Fourier's expansion. For this purpose consider the two curves

$$y = \Psi(x), \qquad y = S_n(x),$$

the second of which we will call the n^{th} approximation curve. It follows at once from the results just obtained that the ordinates of this approximation curve are too large at the points $x_1, x_3, \cdots$, too small at the points $x_2, x_4, \cdots$. The approximation curve therefore has the form of a wavy line which keeps crossing the line Ψ and which reaches its greatest distances from this line (these distances being measured parallel to the axis of y) at the points $x_1, x_2, x_3, \cdots$. Moreover these greatest distances are, for large values of n, approximately equal to their limiting values, $P_1, P_2, P_3, \cdots$. Accordingly, and this is the remarkable phenomenon first noticed by Gibbs,* *the height of these waves does not approach zero as k becomes infinite.* It is true that if we fix our attention on any interval which does not include or reach up to a point of discontinuity of Ψ, the heights of all the waves in this interval approach zero as their limit, since by theorem III of §6 the series converges uniformly in this interval. If, however, we fix on a particular wave determined as the first, second, or in general the k^{th} to the right of the origin, the height of that particular wave as n becomes infinite approaches a finite limit P_k different from zero. At the same time this wave is moving up nearer and nearer to the origin.

Since both $\Psi(x)$ and $S_n(x)$ are odd functions of x, it is clear that the method of approximation to the left of the origin will be essentially the same as to the right. This method of approximation is of course repeated at all the points of discontinuity of Ψ.

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* *Nature*, vol. 59, 1899, p. 606. The same thing was subsequently noticed by Poincaré, ib., vol. 60, p. 52.

This is brought out very clearly, both for the function Ψ and for special cases of other functions which we shall consider presently, in the diagrams constructed by Michelson and Stratton by means of their Harmonic Analyser with which the approximation curves ($n \leqq 80$) for any function which can be represented by a pure sine or a pure cosine series may be constructed mechanically.*

We can now easily generalize from the function Ψ to more general functions along the lines of §7.

In the first place, all we have said concerning Ψ will obviously hold for the function $\Psi_a(x)$ if we remember that the discontinuity of this function, and therefore also the non-vanishing waves of the approximation curves, will be at the point $x = a$ instead of the point $x = 0$.

Considering next the function

$$\frac{\lambda}{\pi} \Psi_a (x),$$

we see that the ordinates of all curves have been changed in a constant ratio. The heights of the waves will therefore also be changed in this ratio, but since the same is true of the magnitude of the finite jump at a, the ratio of the heights of the waves to the magnitude of this jump will be unaltered.

Finally we pass by steps 3) and 4) of §7 to the function $f(x)$ by adding to the function last considered other functions which are continuous at a and whose Fourier's developments converge uniformly throughout the neighborhood of a. This has, on the one hand, no effect on the magnitude of the jump in our function, and, on the other hand, no essential effect when n is large on the heights of the non-vanishing waves of the approximation curve $y = S_n(x)$, since the remainders of the Fourier's series which have been added are uniformly small. Thus we get the following general result:

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* For an account of the machine accompanied by extremely interesting plates the reader is referred to the *Philosophical Magazine*, 5th series, vol. 45, 1898, p. 85. That it would not have been safe to infer the mathematical fact from these diagrams is strikingly shown by a second inference which might have been drawn with about the same degree of certainty from them and which would have been erroneous, namely that at a point where the function developed is continuous and even analytic a wave whose height does not approach zero as its limit may be found; cf. figures 1 and 3 in Plate XII of the article cited. That this appearance is due to some imperfection in the machine becomes evident from Plate XVII where this small wave will be found in the first approximation curve $y = \cos x$.

I. *If $f(x)$ has the period 2π and in any finite interval has no discontinuities other than a finite number of finite jumps, and if it has a derivative which in any finite interval has no discontinuities other than a finite number of finite discontinuities, then if a is any point where $f(x)$ has a finite jump of magnitude D, and if $S_n(x)$ denotes the sum of the first $n+1$ terms in the Fourier's expansion of $f(x)$, the curve $y = S_n(x)$ will for large values of n pass in almost a vertical direction through a point whose abscissa is a and whose ordinate is almost equal to $\frac{1}{2}[f(a+0)+f(a-0)]$. The curve then rises and falls abruptly on the two sides of this point to the neighborhood of the curve $y=f(x)$, and oscillates about this curve lying alternately above and below it. The highest (or lowest)* point of the k^{th} waves to the right and left of a will, for large values of n, lie approximately at the points $a \pm \dfrac{2k\pi}{2n+1}$, and the height of these waves will be approximately $\dfrac{DP_k}{\pi}$.*

Thus, referring to the numerical values given above, we see that the height of the first wave on each side will be about 9% of the magnitude of the discontinuity † while for the subsequent waves the corresponding percentage is about 5%, $3\frac{1}{3}$%, $2\frac{1}{2}$%, 2%, $1\frac{2}{3}$%, $1\frac{2}{5}$% etc.

Gibbs himself did not enunciate his result in such detail as we have done here, nor did he consider any function except the special function Ψ. For this function he made a statement which we generalize as follows:

II. *If $f(x)$ satisfies the conditions of theorem* I, *then as n becomes infinite the approximation curve $y = S_n(x)$ approaches uniformly the continuous curve* ‡ *made up of*

(*a*) *the discontinuous curve $y=f(x)$,*

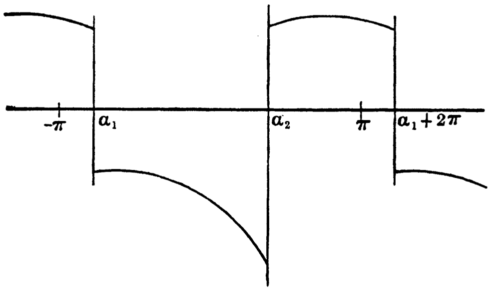
(*b*) *an infinite number of straight lines of finite lengths parallel to the*

* It must be borne in mind that we measure the height of a wave from the curve $y=f(x)$ in a direction parallel to the axis of y.

† More exactly 8.95 %.

‡ That the continuous *curve* $y = S_n(x)$ approaches a *continuous* curve as its limit, while the continuous *function* $S_n(x)$ approaches the *discontinuous* function $f(x)$ as its limit seems at first sight paradoxical, but the paradox disappears when we recall exactly what we mean by saying that one function approaches another. It was the failure to formulate this point clearly which led Michelson to maintain (*Nature*, vol. 58, 1898, p. 544) that Fourier's series of the sort we are here considering really represent continuous functions. The true state of affairs was clearly brought out in the subsequent discussion vols. 58–60, particularly by Love and Gibbs.

axis of y and passing through the points a_1, a_2, $\cdots$ on the axis of x where the discontinuities of $f(x)$ occur. If a is any one of these points, the line in question extends between the two points whose ordinates are



$$f(a-0)+\frac{DP_1}{\pi},\ f(a+0)-\frac{DP_1}{\pi},$$

where D is the magnitude of the jump in $f(x)$ at a, and

$$P_1=\int_\pi^\infty \frac{\sin x}{x}\,dx=-0.2811.$$

The truth of this theorem follows at once from theorem I. It is illustrated by the accompanying figure where the amounts of the jumps at a_1, a_2 are respectively negative and positive. Until Gibbs made his remark it seems to have been supposed that the vertical lines extend merely between the points whose ordinates are $f(a-0)$ and $f(a+0)$.

10. Finite Trigonometric Series.* This subject, which has already been touched upon in §1, is of importance both for its own sake and for its relations to the theory of infinite trigonometric series. Both of these points of view will be kept in mind in the present section.

I. *The $2k+1$ coefficients in the finite trigonometric series*

$$S_k(x)=\tfrac{1}{2}a_0+\sum_1^k(a_n\cos nx+b_n\sin nx) \tag{60}$$

can be determined in such a way that

$$S_k(x_i)=y_i \qquad (i=1,2,\cdots 2k+1), \tag{61}$$

where $y_1, \cdots y_{2k+1}$ are arbitrarily given constants and $x_1, \cdots x_{2k+1}$ are arbitrarily given points in the interval $-\pi<x\leqq\pi$ no two of which are coincident.

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* In connection with this section the reader may consult: Klein, *Auwendung d. Diff.u. Int. Rechnung auf Geometrie*, Leipzig, 1902, pages 139–171.

In order to prove this let us consider the function*

$$(62)\quad \sum_{i=1}^{i=2k+1} y_i \frac{\sin\frac{x-x_1}{2}\cdots\sin\frac{x-x_{i-1}}{2}\sin\frac{x-x_{i+1}}{2}\cdots\sin\frac{x-x_{2k+1}}{2}}{\sin\frac{x_i-x_1}{2}\cdots\sin\frac{x_i-x_{i-1}}{2}\sin\frac{x_i-x_{i+1}}{2}\cdots\sin\frac{x_i-x_{2k+1}}{2}}.$$

None of the quantities in the denominators are zero since $x_i - x_j$ cannot be a multiple of 2π. If we can show that this function (62) can be thrown into the form (60) our theorem will be established, since (62) obviously satisfies conditions (61). We first prove the following

LEMMA:† *A function of the form*

$$(\alpha_1 + \beta_1 \cos x + \gamma_1 \sin x)(\alpha_2 + \beta_2 \cos x + \gamma_2 \sin x)\cdots(\alpha_k + \beta_k \cos x + \gamma_k \sin x)$$

can always be expressed in the form (60).

Since this lemma is true when $k = 1$, it will be established by the principle of mathematical induction if we can show that any expression of the form $(\alpha + \beta \cos x + \gamma \sin x)\, S_{k-1}(x)$ can be reduced to the form (60). Now we may write $\cos x\, S_{k-1}(x)$ in the form:

$$\frac{a_0 \cos x}{2} + \sum_{1}^{k-1}\left(a_n \frac{\cos(n+1)x + \cos(n-1)x}{2} + b_n \frac{\sin(n+1)x + \sin(n-1)x}{2}\right)$$

which is of the form (60). Similarly we see that $\sin x\, S_{k-1}(x)$ may be written in the form (60). Hence $(\alpha + \beta \cos x + \gamma \sin x)\, S_{k-1}(x)$ may be written in the form (60), and our lemma is established.

Going back now to formula (62), we see that, since we may write

$$\sin\frac{x-x_m}{2}\sin\frac{x-x_n}{2} = \alpha + \beta\cos x + \gamma \sin x,$$

and since the numerator of every fraction in (62) contains an even number of factors, every term of (62) can be written as a product of exactly the

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* Cf. Gauss, *Werke*, vol. 3, p. 281.

† From this lemma the first part of the following more general theorem follows at once, while the second part is a consequence of De Moivre's theorem:

Any polynomial of the k^{th} degree in $(\sin x, \cos x)$ *can be written in the form* (60), *and conversely, any function of the form* (60) *can be written as a polynomial in* $(\sin x, \cos x)$ *of degree not higher than* k.

form considered in our lemma, and therefore (62) itself can be written in the form (60). Thus our theorem is proved.

In the observational or experimental sciences, we frequently know that a certain function $f(x)$ is periodic (we will say for simplicity that it has the period 2π), but the values of this function are known, by observation or otherwise, only when x has the values $x_1, x_2, \cdots x_m$, all of which values we may suppose to lie in the interval $-\pi < x \leqq \pi$. It is, then, often desired to get an approximate expression for $f(x)$ in the form (60), our object being of course to make the function $S_k(x)$ coincide as closely as possible with $f(x)$ at the points $x_1, \cdots x_m$. If $m \leqq 2k+1$ we have just seen that $S_k(x)$ can be made to coincide absolutely with $f(x)$ at these points; otherwise we will adopt the principle of the method of least squares, and, writing $y_i = f(x_i)$, we will determine the coefficients in (60) so that

$$I_k = \sum_{i=1}^{i=m} \left\{ y_i - S_k(x_i) \right\}^2$$

is a minimum. This method will in fact apply to all cases, since, when the coefficients in (60) can be so determined that $S_k(x)$ coincides with $f(x)$ at the points x_i, this determination is precisely the one which gives to I_k its minimum value, namely zero.

We will carry through the method here outlined only in a special case which was first treated by Bessel,* namely that in which the points $x_1, x_2, \cdots x_m, x_1 + 2\pi$ are equally spaced, so that $x_{i+1} - x_i = 2\pi/m$. Moreover we will consider merely the case $m \geqq 2k+1$, referring the reader for the treatment of the case† $m < 2k+1$ to pages 337–338 of Bessel's paper.

In order to make I_k a minimum, we follow a method which in all its details is closely parallel to that of pages 83–84 of the present paper. We have

$$\frac{\partial S_k(x_i)}{\partial a_0} = \tfrac{1}{2}, \quad \frac{\partial S_k(x_i)}{\partial a_n} = \cos nx_i, \quad \frac{\partial S_k(x_i)}{\partial b_n} = \sin nx_i.$$

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* *Astronomische Nachrichten*, vol. 6 (1828), p. 333.

† This is the problem of determining the most general expression $S_k(x)$ which takes on arbitrarily given values at less than $2k+1$ points. The solution of this problem must obviously contain undetermined parameters.

Accordingly

$$\frac{\partial I_k}{\partial a_0} = -\sum_1^m y_i + \sum_1^m S_k(x_i),$$

$$\frac{\partial I_k}{\partial a_n} = -2\sum_1^m y_i \cos nx_i + 2\sum_1^m S_k(x_i)\cos nx_i,$$

$$\frac{\partial I_k}{\partial b_n} = -2\sum_1^m y_i \sin nx_i + 2\sum_1^m S_k(x_i)\sin nx_i.$$

In order to reduce these expressions, we make use of the following formulæ, where p is any one of the integers $0, 1, \cdots m-1$:

$$(63) \qquad \sum_{i=1}^{i=m} \cos px_i = \begin{cases} 0 & \text{when } p \neq 0, \\ m & \text{when } p = 0; \end{cases} \qquad \sum_{i=1}^{i=m} \sin px_i = 0.$$

The analytical proof of these formulæ is not difficult, but their truth is at once evident when we regard them as expressing the fact that the centre of gravity of a number of equal particles equally spaced on the circumference of a circle is the centre of this circle.*

From formulæ (63) follow at once the following, in which p, q, $p+q$ are positive integers less than m:

$$(64)\begin{cases} \displaystyle\sum_{i=1}^{i=m} \sin(px_i)\cos(qx_i) = \tfrac{1}{2}\sum_1^m \Big[\sin(p+q)x_i + \sin(p-q)x_i\Big] = 0, \\ \displaystyle\sum_{i=1}^{i=m} \cos(px_i)\cos(qx_i) = \tfrac{1}{2}\sum_1^m \Big[\cos(p-q)x_i + \cos(p+q)x_i\Big] = \begin{cases} 0 & (p \neq q), \\ \frac{m}{2} & (p = q), \end{cases} \\ \displaystyle\sum_{i=1}^{i=m} \sin(px_i)\sin(qx_i) = \tfrac{1}{2}\sum_1^m \Big[\cos(p-q)x_i - \cos(p+q)x_i\Big] = \begin{cases} 0 & (p \neq q), \\ \frac{m}{2} & (p = q). \end{cases} \end{cases}$$

* An exception occurs in the case of one particle. This accounts for the exception noted in the formula when $p = 0$. There would be further exceptions of the same sort whenever p is a multiple of m. These cases we have excluded by assuming $p < m$.

Using these values (63) and (64), we find:

$$\frac{\partial I_k}{\partial a_0} = - \sum_1^m y_i + \tfrac{1}{2} m a_0,$$

$$\frac{\partial I_k}{\partial a_n} = - 2 \sum_1^m y_i \cos n x_i + m a_n,$$

$$\frac{\partial I_k}{\partial b_n} = - 2 \sum_1^m y \sin n x_i + m b_n.$$

Equating these partial derivatives to zero gives us equations which completely determine the coefficients of $S_k(x)$, and that this determination really makes I_k a minimum is evident by a glance at the second derivatives of I_k. The value of this minimum may also be readily computed as on page 85. Our final result may be stated as follows:

II. *If a function $f(x)$ with period 2π takes on the values $y_1, y_2, \cdots y_m$ at the points $x_1, x_2, \cdots x_m$, but is otherwise unknown; and if*

$$x_{i+1} - x_i = \frac{2\pi}{m} \qquad (i = 1, 2, \cdots m-1)$$

and k is a positive integer so small that $m \geqq 2k+1$; then, from the point of view of the method of least squares, the best approximation of the form

$$S_k(x) = \frac{a_0}{2} + \sum_1^k (a_n \cos nx + b_n \sin nx)$$

to $f(x)$ is obtained by giving to the coefficients the values

$$(65) \qquad a_n = \frac{2}{m} \sum_1^m y_i \cos n x_i, \qquad b_n = \frac{2}{m} \sum_1^m y_i \sin n x_i.$$

The minimum value thus obtained for the expression $I_k = \sum_1^m \left\{ y_i - S_k(x_i) \right\}^2$

is

$$\bar{I}_k = \sum_1^m y_i^2 - \frac{m}{2}\left[\frac{a_0^2}{2} + \sum_1^k (a_n^2 + b_n^2)\right].$$

These remarkably elegant formulæ are due to Bessel. In the special case $m = 2k+1$, theorem I tells us that there exists a determination of the coefficients of $S_k(x)$ which makes $I_k = 0$; and, since I_k is never negative, this must

be precisely the determination (65) which, as we have just seen, makes I_k a minimum. Moreover the method we have just used shows that there is only one determination of the coefficients of $S_k(x)$ which makes $I_k = 0$, that is which makes $S_k(x)$ take on the values y_i at the points x_i. Accordingly we get Lagrange's theorem: *

III. *If the points* $x_1, x_2, \cdots x_{2k+1}$ *are so situated that*

$$x_{i+1} - x_i = \frac{2\pi}{2k+1} \qquad (i = 1, 2, \cdots 2k),$$

there exists one and only one determination of the coefficients of $S_k(x)$ *such that*

$$S_k(x_i) = y_i \qquad (i = 1, 2, \cdots 2k+1),$$

and this determination is given by formulæ (65), *in which we must let* $m = 2k + 1$.

We have thus made theorem I very much more precise in the special case in which the points $x_1, \cdots x_{2k+1}$ are evenly distributed throughout the interval $-\pi < x \leqq \pi$. The question now presents itself whether we cannot do something of the same sort in the general case. Before considering this question (cf. theorem VII), we must first establish an important theorem of a different nature due to Sturm.†

IV. *If the constants* a_m *and* b_m *are not both zero, and also* a_k *and* b_k *are not both zero, the function*

$$\phi(x) = \sum_m^k (a_n \cos nx + b_n \sin nx) \qquad (0 \leqq m \leqq k)$$

vanishes and changes sign at at least $2m$ *and at most* $2k$ *distinct points of the interval* $-\pi < x \leqq \pi$.‡

* For Lagrange's method of establishing this result, cf. Byerly's *Fourier's Series, etc.*, pages 30-35. See also Burkhardt's *Bericht*, pages 27–42. Lagrange considered, it is true, not the finite series $S_k(x)$ but the more special series $\sum_1^k b_n \sin nx$, and his result is the special case of that here given in which $x_{k+1} = 0$, and $y_{k+1-i} = -y_{k+1+i}$.

† *Liouville's Journal*, vol. 1 (1836), p. 433. The proof here given is essentially that of Liouville, ib. p 269. Both Sturm and Liouville consider the far more general question of series whose terms instead of being sines and cosines are solutions of certain differential equations.

‡ This theorem is well illustrated by the last two diagrams on plate XVI of the paper by Michelson and Stratton referred to in the foot note on p. 130. See also the curves on pages 51, 52 of Donkin's *Acoustics*.

In order to prove this, let us consider the infinite set of functions

$$\phi_i(x) = \sum_m^k n^{2i}(a_n \cos nx + b_n \sin nx) \qquad (i = \cdots -3, -2, -1, 0, 1, 2, \cdots).$$

All of these functions have the period 2π, and they satisfy the relation

$$\frac{d^2\phi_i(x)}{dx^2} = -\phi_{i+1}(x).$$

Moreover $\phi_0(x) = \phi(x)$. From these facts we readily infer by an application of Rolle's theorem that if $j > i$ the function $\phi_j(x)$ vanishes and changes sign at least as often as $\phi_i(x)$ in the interval $-\pi < x \leqq \pi$. Our theorem will therefore be established if we can show that when i is a large positive integer $\phi_i(x)$ vanishes and changes sign just $2k$ times in the interval $-\pi < x \leqq \pi$, and that when i is a large negative integer $\phi_i(x)$ vanishes and changes sign just $2m$ times. For this purpose we notice that

$$\Phi_i(x) = k^{-i}\phi_i(x) = \sum_m^k \left(\frac{n}{k}\right)^{2i}\left(a_n \cos nx + b_n \sin nx\right)$$

vanishes and changes sign the same number of times as $\phi_i(x)$. Moreover an inspection of the value of $\Phi_i(x)$ shows that

$$\lim_{i=+\infty} \Phi_i(x) = a_k \cos kx + b_k \sin kx,$$

$$\lim_{i=+\infty} \frac{d}{dx}\Phi_i(x) = -ka_k \sin kx + kb_k \cos kx,$$

and that these limits are approached uniformly for all values of x. Accordingly * for large positive values of i, $\Phi_i(x)$ vanishes and changes sign the same number of times as the function $a_k \cos kx + b_k \sin kx$; and, since this function may be written in the form $\sqrt{a_k^2 + b_k^2}\cos k(x - c)$, it vanishes just $2k$ times in the interval $-\pi < x \leqq 2\pi$. That $\phi_i(x)$ vanishes just $2m$ times when i is a large negative integer is seen in the same way by considering the function $m^{-2i}\phi_i(x)$ whose limit is $a_m \cos mx + b_m \sin mx$.

Since the function $\phi(x)$ of the last theorem is an analytic function of x, we may consider the multiplicity of each of its roots.

* The reader should supply the details of the reasoning here. We have to deal with a special case of the theorem proved by the writer in the ANNALS, vol. 6 (1905), p. 61.

V. *The sum of the multiplicities of the roots in the interval* $-\pi < x \leqq \pi$ *of the function* $\phi(x)$ *of theorem* IV *cannot exceed* $2k$.

For combining with Rolle's theorem the fact that at a multiple root of an analytic function its derivative has a root of multiplicity one lower, we see that if the sum of the multiplicities of the roots of $\phi(x)$ in the interval $-\pi < x \leqq \pi$ exceeded $2k$, the same would be true for every function $\phi_i(x)$ where i is positive. But when i is very large, the proof of theorem IV shows that $\phi_i(x)$ has, in this interval, $2k$ roots all of which are simple. Thus our theorem is proved.

A slightly different and often useful form of the theorem just proved is the following:

VI. *If the function*

$$S_k(x) = \frac{a_0}{2} + \sum_1^k (a_n \cos nx + b_n \sin nx)$$

vanishes at more than $2k$ *points in the interval* $-\pi < x \leqq \pi$ *(or if the sum of the orders of its roots in this interval is greater than* $2k$*), all of its coefficients* a_n *and* b_n *are zero.*

We are now in a position to supplement theorem I by the following theorem:

VII. *Only one determination of the coefficients in* (60) *is possible which fulfills conditions* (61).*

For if two such determinations were possible, the difference of the two

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* Explicit expressions for these coefficients may be found by solving by determinants the following system of linear equations:

$$\frac{a_0}{2} + \sum_1^k (a_n \cos nx_i + b_n \sin nx_i) = y_i \qquad (i = 1, 2, \ldots 2k+1).$$

Since we know from theorems I and VII that these equations have one and only one solution, their determinant is not zero. Thus we get incidentally the result:

If $x_1, \ldots x_{2k+1}$ *are any real quantities no two of which differ by an integral multiple of* 2π, *then*

$$\begin{vmatrix} 1 & \cos x_1 & \sin x_1 & \cos 2x_1 & \sin 2x_1 & \ldots & \cos kx_1 & \sin kx_1 \\ 1 & \cos x_2 & \sin x_2 & \cos 2x_2 & \sin 2x_2 & \ldots & \cos kx_2 & \sin kx_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cos x_{2k+1} & \sin x_{2k+1} & \cos 2x_{2k+1} & \sin 2x_{2k+1} & \ldots & \cos kx_{2k+1} & \sin kx_{2k+1} \end{vmatrix} \neq 0.$$

functions $S_k(x)$ thus determined would be a finite trigonometric series whose coefficients, by VI, must all be zero.

Turning now from the theory of finite trigonometric series to more general questions, we are in a position to prove the following theorem, which is essentially due to Liouville:*

VIII. *If $F(x)$ has the period 2π and in any finite interval has only a finite number of discontinuities and is such that $\int |F(x)|dx$ converges when extended over any finite interval; and if the Fourier's constants $a_0, a_1, \cdots a_{k-1}, b_1, \cdots b_{k-1}$ of $F(x)$ are all zero, then $F(x)$ either vanishes at all points where it is continuous, or changes sign*† *at least $2k$ times in the interval $-\pi < x \leqq \pi$.*

To prove this suppose that $F(x)$ changed sign less than $2k$ times in this interval, but did not vanish at all points where it is continuous. Call the points where these changes of sign occur $x_1, x_2, \cdots x_{2m}$, (their number being necessarily even since $F(x)$ is periodic) or, if these changes of sign occur at segments, take one of these x's in each such segment. Now form the function

$$\phi(x) = \tfrac{1}{2} a_0 + \sum_{1}^{m} (a_n \cos nx + \beta_n \sin nx),$$

which vanishes at all these points, while at some point where F is continuous and does not vanish, it has the same sign as $F(x)$. The possibility of thus determining the coefficients of ϕ follows from I. Since ϕ, as thus determined, has $2m$ roots in the interval $-\pi < x \leqq \pi$ but is not identically zero, it can, by VI, vanish nowhere else in this interval, and all these roots must be simple roots. Accordingly ϕ changes sign at each of the points $x_1, \cdots x_{2m}$, and must therefore, except where $F(x)$ vanishes or is discontinuous, have the same sign as $F(x)$ everywhere. Hence

$$\int_{-\pi}^{\pi} F(x)\phi(x)dx > 0.$$

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* *Liouville's Journal*, vol. 1 (1836), p. 264. Liouville tries to infer that if all the Fourier's constants of a continuous function are zero, the function must be identically zero, whereas one can merely infer that the function vanishes an infinite number of times, cf. III, §4.

† We say that F changes sign at a point c when and only when, no matter how small ϵ may be, two points x', x'' can be found such that $c - \epsilon < x' < c \quad x'' < c + \epsilon$ and such that $F(x')$, $F(x'')$ have opposite signs. It should be noticed that the value of F *at* the point c is in no way concerned. F may also change sign not at a point but at a segment cd if it vanishes at all but a finite number of points of this segment, and if, no matter how small ϵ may be, two points x' and x'' can be found such that $c - \epsilon < x' < c < d < x'' < d + \epsilon$ and such that $F(x')$, $F(x'')$ have opposite signs.

Replacing ϕ by its value, this integral reduces to

$$\frac{\pi a_0 a_0}{2} + \pi \sum_1^m (a_n a_n + \beta_n b_n) = 0.$$

Thus the assumption $m < k$ has led to a contradiction and our theorem is established.

The theorem just proved may be applied to a function $f(x)$ none of whose Fourier's constants are assumed to vanish by taking as the function $F(x)$ the difference between $f(x)$ and the sum of the first $k + 1$ terms of the Fourier's development of this function. We thus get the result:

IX. *If $f(x)$ has the period 2π and in any finite interval has only a finite number of discontinuities, and is such that $\int |f(x)| dx$ converges when extended over any finite interval; and $S_k(x)$ is the sum of the first $k + 1$ terms of the Fourier's development of $f(x)$, then the approximation curve $y = S_k(x)$ crosses* the curve $y = f(x)$ at least $2k + 2$ times in the interval $-\pi < x \leqq \pi$.*

It should be noticed that the conditions of this theorem do not imply the convergence of the Fourier's development of $f(x)$, but that the theorem applies to the approximate representations of $f(x)$ considered in §1. This theorem is well illustrated by the diagrams on pages 63, 64 of Byerly's *Elementary Treatise on Fourier's Series, etc.*, as well as by the diagrams by Michelson and Stratton, referred to on page 130.

11. Dirichlet's Integrals. In order to penetrate more deeply into the theory of the convergence of Fourier's series we now turn to the kind of consideration first successfully applied to this question by Dirichlet.†

I. *If in the interval $g \leqq \beta \leqq h$ the function $F(\beta)$ has only a finite number of discontinuities and $\int_g^h |F(\beta)| d\beta$ converges, then*

$$\lim_{k=+\infty} \int_g^h F(\beta) \sin k\beta \, d\beta = 0.‡$$

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* The curves may cross each other without cutting at points where $f(x)$ is discontinuous.

† *Crelle's Journal*, vol. 4 (1829), p. 157.

‡ This theorem was stated by Dirichlet in nearly as general a form as this in a letter to Gauss dated 1853 (Dirichlet's *Collected Works*, vol. 2, p. 386), but in the proof there given he tacitly makes further restrictions on his function.

We prove this theorem first in the special case in which F is continuous throughout the interval $g \leqq \beta \leqq h$.*

Let us divide this interval into $r = 2^m$ equal parts by the points $a_0 = g,\ a_1,\ a_2,\ \cdots a_{r-1},\ a_r = h$; and write

$$\begin{aligned}\int_g^h & F(\beta)\sin k\beta d\beta \\ &= \sum_0^{r-1}\left\{\int_{a_i}^{a_{i+1}} F(a_i)\sin k\beta d\beta + \int_{a_i}^{a_{i+1}}[F(\beta)-F(a_i)]\sin k\beta d\beta\right\} \\ &= \sum_0^{r-1}\left\{F(a_i)\frac{\cos ka_i - \cos ka_{i+1}}{k} + \int_{a_i}^{a_{i+1}}[F(\beta)-F(a_i)]\sin k\beta d\beta\right\}.\end{aligned}$$

Accordingly, if we denote by M the upper limit of $|F(\beta)|$ in the interval gh, and by η the greatest oscillation† of F in any one of the intervals a_0a_1, a_1a_2, $\cdots a_{r-1}a_r$, we have

$$(66)\quad \left|\int_g^h F(\beta)\sin k\beta d\beta\right| \leqq \sum_0^{r-1}\left\{\frac{2M}{k} + \eta(a_{i+1}-a_i)\right\} = \frac{2Mr}{k} + \eta(h-g).$$

So far we have not restricted the value of the integer $r = 2^m$ on which η depends. Let us now choose it as the largest integral power of 2 which does not exceed $\sqrt{k}$. Then $2Mr/k \leqq 2M/\sqrt{k}$, which approaches zero when $k = \infty$. Since r becomes infinite with k, η approaches zero as k becomes infinite.‡ Hence the whole expression (66) approaches zero, and the special case of I is proved.

Turning now to the general case, suppose that F has n discontinuities in the interval $g \leqq \beta \leqq h$. Our theorem will be proved if we can show that however small the positive quantity ϵ may be, a positive constant K can be found such that

$$(67)\qquad \left|\int_g^h F(\beta)\sin k\beta d\beta\right| < \epsilon \qquad \text{when } k > K.$$

* The proof in the more general case in which $F(\beta)$ is finite and integrable was given by Riemann (1854), *Ges. Werke*, 2d. ed., p. 254. We follow Staeckel, *Leipziger Berichte*, vol. 53 (1901), p. 148.

† By the oscillation of a function in an interval is understood the difference between its upper and lower limits in that interval.

‡ Since, when we divide an interval into 2^m equal parts, the integer m can be taken so large that the oscillations of a given function (which is supposed to be continuous within and at the extremities of the interval) are less, in all these parts, than an arbitrarily given positive constant. See Picard's *Traité d'analyse*, vol. 1, p. 3.

Surround each of the n points of discontinuity of F by a neighborhood so small that $\int |F(\beta)| d\beta < \epsilon/(2n)$, the integral being extended over any one of these neighborhoods. It follows that for all values of k the sum of the n integrals $\int F(\beta) \sin k\beta d\beta$, each extended over one of these neighborhoods, is, in absolute value, less than $\frac{1}{2}\epsilon$. By the special case of I which has been proved, $\int F(\beta) \sin k\beta d\beta$, when extended over a fixed interval which does not reach up to a point of discontinuity of F, approaches zero when $k = \infty$. Accordingly the same will be true of the sum of the integrals of this form extended over all parts of the interval gh except the neighborhoods, above mentioned, of the points of discontinuity. In other words a constant K can be determined such that when $k > K$ the sum of these integrals is, in absolute value, less than $\frac{1}{2}\epsilon$. Thus by first taking the neighborhoods of the points of discontinuity small enough, and then choosing K large enough, we have satisfied (67).

II. *If $F(\beta)$ has only a finite number of discontinuities in the interval $0 \leqq \beta \leqq h$, and the limit $F(+0)$ exists, and if the integral*

$$\int_0^h \frac{|F(\beta) - F(+0)|}{\beta} d\beta \tag{68}$$

converges, then

$$\lim_{k=+\infty} \int_0^h F(\beta) \frac{\sin k\beta}{\beta} d\beta = \tfrac{1}{2}\pi F(+0). \tag{69}$$

This theorem, which was essentially known to Dirichlet in 1853,* though first published by Dini† follows from I if we write

$$\int_0^h F(\beta) \frac{\sin k\beta}{\beta} d\beta = \int_0^h F(+0) \frac{\sin k\beta}{\beta} d\beta + \int_0^h \frac{F(\beta) - F(+0)}{\beta} \sin k\beta \, d\beta.$$

By I, the last of these integrals approaches zero when $k = \infty$. Letting $z = k\beta$, the other integral may be written

$$F(+0) \int_0^{kh} \frac{\sin z}{z} dz,$$

* See the letter to Gauss referred to above.

† *Serie di Fourier*, Pisa, 1880, p. 87.

from which II follows by the well known formula*

$$\int_0^{+\infty} \frac{\sin z}{z}\, dz = \frac{\pi}{2}.$$

We shall find it convenient to lay down the following definition:

CONDITIONS (A_+) AND (A_-). *A function $F(\beta)$ shall be said to satisfy condition (A_+) at a point c if it is continuous throughout a certain interval $c < \beta \leqq c + \delta$, if the limit $F(c + 0)$ exists, and the integral*

$$(70) \qquad \int_c^{c+\delta} \frac{|F(\beta) - F(c+0)|}{\beta - c}\, d\beta$$

converges. It shall be said to satisfy condition (A_-) at c if it is continuous throughout a certain interval $c - \delta \leqq \beta < c$, if the limit $F(c - 0)$ exists, and the integral

$$(71) \qquad \int_{c-\delta}^{c} \frac{|F(\beta) - F(c-0)|}{\beta - c}\, d\beta$$

converges.

It will be noticed that the fractions in (70) and (71) are simply the forward and backward difference quotients of F at c. Accordingly, since these integrals must converge when their integrands approach finite limits at c, we may say:

III. *If F has a finite forward derivative at c, (70) converges. If f has a finite backward derivative at c, (71) converges.*

Or more generally:

IV. *Integrals (70) and (71) respectively will converge if F satisfies the first or second of the inequalities*

$$|F(\beta) - F(c+0)| < K(\beta - c)^a \qquad (c < \beta \leqq c + \delta),$$

$$|F(\beta) - F(c-0)| < K(c - \beta)^a \qquad (c - \delta \leqq \beta < c),$$

where K and a are positive constants.

Theorem II may now be thrown into the following convenient form:

V. *If in the interval $g \leqq \beta \leqq h$ the function $F(\beta)$ has only a finite number of discontinuities, and $\int_g^h |F(\beta)|\, d\beta$ converges, and if c is a point ($g < c < h$) at which F satisfies conditions (A_+) and (A_-), then*

$$(72) \qquad \lim_{k=+\infty} \int_g^h F(\beta) \frac{\sin k(\beta - c)}{\beta - c}\, d\beta = \tfrac{1}{2}\pi \left\{ F(c+0) + F(c-0) \right\}.$$

* See for instance Osgood's *Funktionentheorie*, vol. 1, p. 246.

To prove this, write the integral in (72) as the sum of the integrals from g to c and from c to h, and make in the first the change of variable $\beta' = c - \beta$, in the second $\beta'' = \beta - c$. Thus both integrals are reduced to the form (69), and our theorem follows at once.

We add the following theorem which we shall need in the next section.

VI. *If $F_1(\beta)$ and $F_2(\beta)$ satisfy condition (A_+), [or (A_-)] at c, their sum and product both satisfy this condition at c.*

That this is true for the sum is obvious. To prove it for the product $F(\beta) = F_1(\beta)F_2(\beta)$, write

$$\frac{F(\beta) - F(c+0)}{\beta - c} = F_1(\beta)\,\frac{F_2(\beta) - F_2(c+0)}{\beta - c} + F_2(c+0)\,\frac{F_1(\beta) - F_1(c+0)}{\beta - c}\,.$$

The part of the theorem referring to (A_+) follows from this equation. The part referring to (A_-) may be proved in a similar way.

12. Dirichlet's Second Sufficient Condition for the Convergence of Fourier's Series (Dini). It will be convenient to lay down the following definition:

CONDITION B. *A function $f(x)$ shall be said to satisfy condition (B) if*

1) *it has the period 2π;*

2) *in the interval $-\pi \leqq x \leqq \pi$ it has at most a finite number of discontinuities;*

3) *the integral* $\int_{-\pi}^{\pi} |f(x)|\,dx$ *converges.*

These conditions are sufficient to secure the existence of the Fourier's constants of $f(x)$:*

$$a_\nu = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\beta)\cos\nu\beta\,d\beta, \qquad b_\nu = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\beta)\sin\nu\beta\,d\beta.$$

They are, however, not enough to ensure the convergence of the Fourier's

* Instead of condition 3), it would be sufficient for this purpose to assume that $\int_{-\pi}^{\pi} f(x)dx$ converges, as is readily seen by an integration by parts.

development of $f(x)$.* To secure this convergence further conditions must be imposed on $f(x)$ as, for instance, in the following theorem:

I. *The Fourier s development of a function $f(x)$ which satisfies condition (B) converges to the value $\frac{1}{2}[f(x+0)+f(x-0)]$ at every point where $f(x)$ satisfies conditions (A_+) and (A_-).*

To prove this, consider the sum of the first $n+1$ terms of the Fourier's development of $f(x)$:

$$S_n(x) = \tfrac{1}{2}a_0 + \sum_1^n (a_\nu \cos \nu x + b_\nu \sin \nu x).$$

Substituting here for a_ν and b_ν their values written above, we have

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\beta)\left\{\tfrac{1}{2} + \cos(\beta - x) + \cos 2(\beta - x) + \cdots + \cos n(\beta - x)\right\} d\beta,$$

which by a well known trigonometrical identity (formula (46) above) reduces to

$$(73) \qquad S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\beta)\frac{\sin[(n+\frac{1}{2})(\beta - x)]}{2\sin\frac{1}{2}(\beta - x)}\, d\beta.$$

Since the integrand here is a periodic function of β of period 2π, the limits of integration may be changed to g and $g + 2\pi$, where g may be chosen at pleasure. Let us take g in such a way that the point x we wish to consider lies in the interval $g < x < g + 2\pi$. We may now write (73) in the form

$$(74) \qquad S_n(x) = \frac{1}{\pi}\int_g^{g+2\pi} F(\beta)\, \frac{\sin[(n+\frac{1}{2})(\beta - x)]}{\beta - x}\, d\beta,$$

where $F(\beta) = f(\beta)\cdot f_1(\beta)$ and

$$f_1(\beta) = \frac{\beta - x}{2\sin\frac{1}{2}(\beta - x)} \quad (\beta \neq x), \qquad\qquad f_1(x) = 1.$$

It is clear that $f_1(\beta)$ satisfies conditions (A_+) and (A_-) at x since it is continuous there and has a finite derivative. Accordingly, if $f(\beta)$ satisfies con-

* This fact has been established by the formation of functions whose Fourier's developments diverge at certain points. In fact functions have been formed which are everywhere continuous but whose Fourier's developments diverge at points everywhere dense. It is not known whether continuous functions exist whose Fourier's developments diverge at *all* points. The most recent work on the questions here referred to is by Hobson, *Proc. Lond. Math. Soc.*, ser. 2, vol. 3 (1905), p. 48. References to earlier work are there given.

ditions (A_+) and (A_-) at x, the same will (by VI §11) be true of $F(\beta)$, and $F(x+0) = f(x+0)$, $F(x-0) = f(x-0)$. Hence (by V §11) $S_n(x)$ approaches $\frac{1}{2}[f(x+0) + f(x-0)]$ as n becomes infinite, and our theorem is proved.

A reference to theorem III of §11 shows that the convergence of the Fourier's series at a point x will be secured if in addition to condition (B) we demand that $f(x)$ have a finite forward and a finite backward derivative at that point. Our theorem also includes many cases — all that are likely to occur in practice — of infinite derivatives, namely (IV §11) all those in which $f(\beta)$ has the form $C + (\beta - x)^a \phi(\beta)$ where $0 < a < 1$ and ϕ is finite at x.

There is an essential difference between the condition for convergence of Fourier's series obtained in this section and those obtained earlier in this paper. Condition (B) or a still narrower condition has been imposed in all cases in order to secure the existence of the Fourier's constants. The further condition imposed in §§5, 7 was that f have a derivative which in any finite interval has at most a finite number of finite discontinuities. This condition imposes a restriction on $f(x)$ throughout the whole interval $-\pi < x < \pi$, whereas conditions (A_+) and (A_-), which we now impose, restrict the nature of the function only in the immediate neighborhood of the point at which the convergence of the series is to be considered. In fact we may readily deduce from formula (73) in combination with theorem I §11 the following result, which, in a still more general form, is due to Riemann:

II. *The convergence at a point x of the Fourier's development of a function which satisfies condition (B) depends merely on the nature of the function in the immediate neighborhood of x.*

13. Dirichlet's First Sufficient Condition for the Convergence of Fourier's Series (Jordan). We begin by recalling the Second Law of the Mean which, in its simplest form, is as follows:*

THE SECOND LAW OF THE MEAN. *If in the interval $g < \beta < h$ the two functions $\phi(\beta)$ and $\psi(\beta)$ are finite and continuous while the first of them is nowhere negative in this interval and never decreases as β increases, then*

$$\int_g^h \phi(\beta)\psi(\beta)d\beta = \phi(h-0)\int_\xi^h \psi(\beta)d\beta \qquad (g < \xi < h).$$

* Bonnet's original proof is reproduced by Picard *Traité d'analyse*, 2d. ed., vol. 1, p. 8. See also C. Neumann *Ueber die nach Kreis-, Kugel- und Cylinder-Functionen fortschreitenden Entwickelungen*, p. 34.

We follow Bonnet* in using this theorem to deduce the result which forms the starting point of Dirichlet's classic paper:

I. *If in the interval* $0 < \beta < h$ *the function* $F(\beta)$ *is finite, continuous, and nowhere decreasing, then*

$$\lim_{k=+\infty} \int_0^h F(\beta)\frac{\sin k\beta}{\beta}\,d\beta = \tfrac{1}{2}\pi F(+0).$$

As in the proof of II §11, this theorem will be proved if we can establish the formula

$$\lim_{k=+\infty} \int_0^h \left\{F(\beta) - F(+0)\right\}\frac{\sin k\beta}{\beta}\,d\beta = 0. \tag{74}$$

Let b be a constant satisfying the inequality $0 < b < h$. By the second law of the mean,

$$\int_0^b \left\{F(\beta) - F(+0)\right\}\frac{\sin k\beta}{\beta}\,d\beta = \left\{F(b) - F(+0)\right\}\int_\xi^b \frac{\sin k\beta}{\beta}\,d\beta \quad (0<\xi<b). \tag{75}$$

Since $\int_0^\infty \frac{\sin z}{z}dz$ is convergent, there exists a positive constant M such that, for all positive values of p, $\left|\int_0^p \frac{\sin z}{z}\,dz\right| < M$. Accordingly

$$\left|\int_\xi^b \frac{\sin k\beta}{\beta}\,d\beta\right| = \left|\int^{kb} \frac{\sin z}{z}\,dz - \int_0^{k\xi}\frac{\sin z}{z}\,dz\right| < 2M.$$

Hence by taking b so small that $|F(b) - F(+0)| < \epsilon/(4M)$, we can make the integral (75) less in absolute value than $\frac{1}{2}\epsilon$ for all values of k.

On the other hand we have (by I §11), b being regarded as fixed,

$$\lim_{k=+\infty} \int_b^h \left\{F(\beta) - F(+0)\right\}\frac{\sin k\beta}{\beta}\,d\beta = 0.$$

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* *Mémoires de l'Académie de Belgique*, vol. 23 (1850), p. 16.

By first choosing b sufficiently small, and then taking K sufficiently large, we thus have, when $k > K$,

$$\left|\int_0^h \left\{F(\beta) - F(+0)\right\} \frac{\sin k\beta}{\beta}\,d\beta\right| \leqq \left|\int_0^b \left\{F(\beta) - F(+0)\right\} \frac{\sin k\beta}{\beta}\,d\beta\right| + \left|\int_b^h \left\{F(\beta) - F(+0)\right\} \frac{\sin k\beta}{\beta}\,d\beta\right| < \epsilon.$$

This inequality being equivalent to (74), our theorem is proved.

We now proceed to generalize theorem I. For this purpose we lay down the following

DEFINITION. *A continuous function $f(\beta)$ is said to have limited variation in an interval $g < \beta < h$ if it can be written in the form*

$$f(\beta) = f_1(\beta) - f_2(\beta), \tag{76}$$

where throughout the interval f_1 and f_2 are continuous, finite, and positive, and never decrease.

The theory of these functions as developed by Jordan* will not be necessary for our purposes, but only the following facts which follow from (76).

In the first place the limits $f(g+0)$ and $f(h-0)$ exist.

Secondly the sum and product of two continuous functions with limited variation are continuous functions with limited variation.

Finally any continuous function which in a given interval is finite and never decreases (or never increases) has limited variation there; for in the first case it can be written in the form $[c + f(\beta)] - c$, in the second in the form $c - [c - f(\beta)]$, which, if the constant c is properly chosen, have the form (76).

CONDITIONS (C_+) AND (C_-). *A function $F(\beta)$ shall be said to satisfy condition (C_+) [or (C_-)] at c if throughout a certain interval $c < x < c + \delta$ [or $c - \delta < x < c$] it is continuous and has limited variation.*†

* *Cours d'analyse*, vol. 1, p. 54. See also Pierpont's *Functions of Real Variables*, p. 349. Jordan's starting point (C. R. 1881) was precisely that indicated in the text. The restriction we have made that the function be continuous is not necessary and was not made by Jordan.

† The functions which most frequently occur in practice satisfy, at all points where they remain finite, both conditions (A) and conditions (C). These conditions are, however, not coextensive, nor does either include the other, as the two functions $\beta \sin(1/\beta)$ and $1/\log|\beta|$ show. The first of these satisfies conditions (A) at the point $\beta = 0$ but not conditions (C) (cf. Pierpont, l. c., p. 352), while the second satisfies conditions (C) but not conditions (A) at this point.

A special case in which these conditions are satisfied is the following:

II. *If a function has only a finite number of discontinuities in an interval, it satisfies conditions* (C_+) *and* (C_-) *at any point of this interval where it remains finite and in whose neighborhood it does not have an infinite number of maxima and minima.*

We now see, by using (76), that I still holds if we merely require $F(\beta)$ to be continuous and to have limited variation in the interval $0 < \beta \leqq h$. We may even go a step further and require merely that $F(\beta)$ satisfy condition (C_+) at the point $\beta = 0$, that it have only a finite number of discontinuities, and that $\int_g^h |F(\beta)|\, d\beta$ converge, as we see by breaking up the interval $0h$ into the two intervals 0δ and δh and applying theorem I §11 to the second of these intervals.

Theorem I thus generalized admits of immediate extention to the following form:

III. *Theorem* V §11 *still holds if the requirement that* F *satisfy conditions* (A_+) *and* (A_-) *at* c *be replaced by the requirement that it satisfy conditions* (C_+) *and* (C_-) *there, or if either of these changes be made.*

From this we pass, exactly as in §12, to the final result:

IV. *The Fourier's development of a function* $f(x)$ *which satisfies condition* (B) *converges to the value* $\frac{1}{2}[f(x+0) + f(x-0)]$ *at every point where* $f(x)$ *satisfies either condition* (A_+) *or* (C_+) *and also either condition* (A_-) *or* (C_-). *In particular it converges to this value at every point in whose neighborhood* $f(x)$ *does not have an infinite number of maxima and minima.*

It follows that if $f(x)$ does not have an infinite number of maxima and minima in the interval $-\pi < x < \pi$, its Fourier's development will converge to the value $\frac{1}{2}[f(x+0) + f(x-0)]$ at every point where $f(x)$ is finite; a result which includes that originally established by Dirichlet.

14. Conclusion. Certain important aspects of the theory of trigonometric series which, for want of space, we have been unable to treat at length will now be briefly referred to.

The question of obtaining as broad sufficient conditions as possible for the convergence of Fourier's series occupied the attention of many mathematicians during the last half of the nineteenth century. The reader wishing to penetrate into this field of research may turn to an important article by Lebesgue, *Math. Ann.*, vol. 61 (1905), p. 251.

Simultaneously with these investigations, researches of a more general

character concerning trigonometric series appeared during the years immediately succeeding the publication in 1867 of Riemann's great paper on trigonometric series. From these we mention first Heine's results concerning the uniform convergence of Fourier's series,* and then the following two important theorems first completely proved by G. Cantor.

I. *If a trigonometric series converges at all points of an interval* $A < x < B$, *its coefficients* a_n *and* b_n *approach zero when* $n = \infty$.†

II. *If two trigonometric series converge to the same values at all but a finite number of points of the interval* $-\pi < x < \pi$, *their corresponding coefficients are equal.*‡

Recently a new turn (related, however, to Poisson's method of convergence factors) has been given to the theory of Fourier's series by the investigations of la Vallée Poussin,§ Hurwitz,‖ and Fejér,¶ in which it is shown how *even though the Fourier's development of a function diverge*, the series may still be used to advantage. The work of the first two of these mathematicians has already been touched upon in this paper (pages 107, 117, 118). Fejér's main result is the following:

III. *If* $f(x)$ *satisfies condition* (B) *of* §13,** *and if* $S_k(x)$ *denotes the sum of the first* $k + 1$ *terms of its Fourier's development, then*

$$\lim_{n=\infty}\left\{\frac{S_0 + S_1(x) + \cdots + S_n(x)}{n+1}\right\} = \tfrac{1}{2}\left\{f(x+0) + f(x-0)\right\}$$

at every point where $f(x)$ *is continuous or has a finite jump, and this limit is approached uniformly throughout any interval* $A \leqq x \leqq B$ *where* $f(x)$ *is continuous.*

It is of considerable practical importance to be able to get quickly the numerical values of the first few Fourier's constants of a given function. Special instruments, known as Harmonic Analysers, have been devised for

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* See Picard, *Traité d'analyse*, 2d ed., vol. 1, p. 256.

† *Math. Ann.*, vol. 4 (1871), p. 139. French translation in *Acta. Math.*, vol. 2, p. 329. If, in this theorem, we assume the convergence to be uniform, the proof becomes very simple. Cf. Heine, *Crelle's Journal*, vol. 71 (1870), p. 357.

‡ See Picard, *Traité d'analyse*, 2d ed., vol. 1, p. 259.

§ See foot-note p. 107.

‖ See foot-note p. 107. Also *Math. Ann.*, vol. 57 (1903), p. 425.

¶ *Math. Ann.*, vol. 58 (1904), p. 51.

** Fejér's condition is much broader than this.

this purpose.* Such instruments, however, being expensive, other methods are in more common use. Cf. Runge, *Theorie und Praxis der Reihen*, p. 147–164, and *Zeitschrift für Math. u. Physik*, vol. 52 (1905), p. 117; Lyle, *Phil. Mag.* ser. 6, vol. 11 (1906), p. 25; and, for numerous further references, S. P. Thompson, *Proc. Phys. Soc. of London*, vol. 19 (1905), p. 443.

Finally we refer to Cauchy's application of his method of residues to prove the convergence of Fourier's series.† This method, as Cauchy pointed out, is applicable to many other similar expansions. For such generalizations (made partly by this and partly by other methods) we refer to:

Dini, *Serie di Fourier.*

C. Neumann, *Ueber die nach Kreis-, Kugel- und Cylinder-Functionen fortschreitenden Entwickelungen.*

Jordan, *Cours d'analyse*, vol. 2, chap. 4.

Kneser, *Math. Ann.*, vol. 58 (1904), p. 81.

Dixon, *Proc. Lond. Math. Soc.*, vol. 3 (1905), p. 83.

Hilbert, *Göttinger Nachrichten*, 1904, pp. 49, 213.

Schmidt, *Entwickelung willkürlicher Funktionen nach Systemen vorgeschriebener*, Dissertation, Göttingen, 1905.

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*Cf. p. 130. An analyser constructed by Coradi is described on p. 155 of Klein's *Anwendung d. Diff. u. Int. Rechnung auf Geometrie.*

† See Picard, *Traité d'analyse*, vol. 2, chap. 6 §II.